

Probing $\mathcal{N} = 4$ SYM with surface operators

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ABSTRACT: In this paper we study surface operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We compute surface operator observables, such as the expectation value of surface operators, the correlation function of surface operators with local operators, and the correlation function of surface operators with Wilson and 't Hooft loops. The calculations are performed using three different realizations of surface operators, corresponding respectively to the gauge theory path integral definition, the probe brane description in $AdS_5 \times S^5$ and the “bubbling” supergravity description of surface operators. We find remarkable agreement between the different calculations performed using the three different realizations.

KEYWORDS: AdS-CFT Correspondence, Brane Dynamics in Gauge Theories.

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1. Introduction and summary

In gauge theories, nonlocal operators can be understood as operators that insert a probe into the theory. Well known are the Wilson and 't Hooft operators, which are characterized geometrically by a curve in spacetime. A Wilson loop inserts into the theory a point-like charged particle while an 't Hooft operator inserts a magnetic monopole. These operators play an important role as order parameters of various phases of gauge theory.

A surface operator on the other hand is characterized geometrically by a surface Σ in spacetime. This surface corresponds to the worldsheet spanned by a probe string that the surface operator inserts in spacetime. A surface operator \mathcal{O}_Σ measures the response of the external string with which the field theory is probed, and therefore these operators may enlarge the list of known order parameters in gauge theories.

In $\mathcal{N} = 4$ SYM two types of supersymmetric surface operators have been studied. One is akin to the 't Hooft operator in that it is an operator of disorder type, characterized by a certain supersymmetric vortex-like singularity near a surface Σ [1]. One can also construct a supersymmetric surface operator of order type [2], akin to a Wilson loop operator, which inserts into the $\mathcal{N} = 4$ SYM path integral the WZW action integrated over Σ together with an associated Chern-Simons term, where the WZW action is constructed out of the $\mathcal{N} = 4$ SYM gauge field along Σ (see also the work in [3]).¹

In this paper we go beyond identifying and classifying these operators and perform three classes of computations with the surface operators \mathcal{O}_Σ . We first study properties associated to the surface operator \mathcal{O}_Σ itself, namely evaluating its expectation value. Then we consider the interaction of a surface operator \mathcal{O}_Σ with local operators, including chiral primary operators and the stress-energy tensor. And then we analyze the interaction between a surface operator \mathcal{O}_Σ and other non-local operators in the gauge theory — certain supersymmetric Wilson loops and 't Hooft loops. These correlators exhibit a rich dependence on the parameters that a surface operator \mathcal{O}_Σ is characterized by, which we review in section 2.1.

We study the disorder surface operators \mathcal{O}_Σ of [1] by using three alternative realizations of them. In the first one we use the gauge theory definition in terms of supersymmetric codimension two singularities in the $\mathcal{N} = 4$ SYM path integral [1]. In the second, a surface operator \mathcal{O}_Σ is realized in terms of a configuration of D3-branes ending on the boundary of $AdS_5 \times S^5$ along the surface Σ , which realizes the boundary conditions corresponding to a surface operator \mathcal{O}_Σ in the gauge theory living on the boundary. The third description of surface operators \mathcal{O}_Σ is in terms of “bubbling” geometries — ten dimensional solutions of Type IIB supergravity which are asymptotically $AdS_5 \times S^5$. They were proposed in [7] as the gravitational description of surface operators and are related to the solutions of Lin, Lunin and Maldacena (LLM) [8, 9] by analytic continuation.

While these correlation functions are computed using a variety of techniques, ranging from purely field theoretical to string theoretic calculations using strings, branes and supergravity solutions, the results are very coherent. A rather complete picture arises that allows us to study the properties of surface operators in $\mathcal{N} = 4$ SYM in different regimes of the gauge theory parameter space. It is quite remarkable that the results of our computations computed in various regimes of $\mathcal{N} = 4$ SYM show agreement with each other. The computations performed using semiclassical techniques in the gauge theory agree with the computations we perform using a bulk dual string theory computation.

The three realizations of a surface operator \mathcal{O}_Σ are valid in different regimes of the gauge theory parameter space. The semiclassical gauge theory description is appropriate at

¹Disorder surface operators with higher pole singularities have been constructed in [4] while rigid disorder surface operators have been studied in [5]. Disorder surface operators have also been considered in [6].

weak 't Hooft coupling $\lambda \ll 1$. The probe D-brane description of a surface operator is valid when $N \gg 1$, $\lambda \gg 1$ and when the number of D3-branes making up the corresponding surface operator is small. Finally, the supergravity description in terms of “bubbling” geometries is appropriate when $N \gg 1$, $\lambda \gg 1$ and when the surface operator \mathcal{O}_Σ is characterized in the probe description by a large number of D3-branes, in which case the backreaction of the D3 branes cannot be neglected. Nevertheless, surprisingly, we obtain quantitative agreement when the the same computation involving surface operators is performed using the different realizations of \mathcal{O}_Σ .

Summary of results. Before getting into the details of the calculations in the following sections, let us elaborate a bit on the quantities we compute in this paper.

The first quantity we calculate is the expectation value of the surface operator \mathcal{O}_Σ . In this paper we consider maximally supersymmetric surface operators \mathcal{O}_Σ , which can have the geometry of the plane or of a sphere, that is $\Sigma = R^2$ or S^2 . Naively, one may think that in a conformal field theory like $\mathcal{N} = 4$ SYM, that the expectation value of \mathcal{O}_Σ cannot depend on the radius a of the S^2 on which \mathcal{O}_Σ is supported. However, just as for local operators, surface operators can have conformal anomalies which introduce a non-trivial scale-dependence. Conformal invariance, however, restricts the functional form of the expectation value to be

$$\langle \mathcal{O} \rangle = C a^\Delta, \quad (1.1)$$

where in principle both C and Δ can depend on the coupling constant of the theory.² We have calculated these quantities in the semiclassical approximation of the gauge theory and also in the probe approximation, where a surface operator \mathcal{O}_Σ is described by a configuration of D3 branes. In both regimes we find that

$$C = 1, \quad \Delta = 0. \quad (1.2)$$

This result is rather different from the one obtained for surface observables in the six-dimensional $(0,2)$ theory dual to string theory on $AdS_7 \times S^4$, where the dimension Δ is non-zero [10–14].

We then calculate the correlation function between a surface operator \mathcal{O}_Σ and local operators \mathcal{O} in $\mathcal{N} = 4$ SYM. We are able to calculate these correlation functions using the three different realizations of surface operators, one using gauge theory, one using probe D3-branes and one using the “bubbling” supergravity solutions. We find remarkable agreement between these correlators computed in the three different regimes.

Such two-point functions are obvious quantities to study since local operators serve as natural probes of surface operators \mathcal{O}_Σ , and it is interesting to unravel how they interact with them. Moreover, when the surface operator \mathcal{O}_Σ is supported on an S^2 , it can be approximated at large distances by a linear combination of local operators [10]. In other

²In the case of local operators one can determine the dimension by calculating a 2-point function. We would like to point out, to avoid confusion, that the analogous quantity here is the vacuum expectation value of a *single* sphere, not two. In both cases there is a single length-scale, the distance between the points or the radius.

words, there is an operator product expansion of a surface operator \mathcal{O}_Σ in terms of local ones \mathcal{O} . The correlators of the surface operator \mathcal{O}_Σ with the local operators \mathcal{O} determine the coefficients in this expansion.

One particularly interesting local operator is the energy-momentum tensor T_{mn} . The correlator between T_{mn} and a local operator \mathcal{O} is proportional to the conformal dimension of the operator \mathcal{O} . When we calculate the correlator between T_{mn} and a surface operator \mathcal{O}_Σ we find a non-zero answer. This is not in contradiction with the already stated result that there is no radius a dependence in the vacuum expectation value of a surface operator \mathcal{O}_Σ . As we summarize in section 2.2, a surface operator \mathcal{O}_Σ has three different possible conformal anomalies and the two-point function of a surface operator with the energy-momentum tensor may calculate a different combination of these three anomaly coefficients.

We then calculate the correlation function between a surface operator \mathcal{O}_Σ and Wilson and t'Hooft loop operators in $\mathcal{N} = 4$ SYM. We are also able to calculate these correlation functions using the three different realizations of surface operators, one using gauge theory, one using probe D3-branes and one using the “bubbling” supergravity solutions. We also find remarkable agreement between these correlators computed in the three different regimes.

The geometrical properties of a correlator between a surface operator \mathcal{O}_Σ supported on a surface Σ in spacetime and a loop operator supported on a curve C in spacetime are quite interesting, since in four dimensions a surface Σ can be linked by a curve C . The correlators of \mathcal{O}_Σ with Wilson and 't Hooft loops exhibit rich dependence on all the parameters that a surface operator \mathcal{O}_Σ is characterized by.

The Wilson loop we consider was constructed in [15] and preserves eight supercharges. This Wilson loop links the surface operator \mathcal{O}_Σ and depends on two parameters (θ_0, ψ_0) which encode the coupling of the six scalars in $\mathcal{N} = 4$ SYM to the loop. It has been argued in [15] that the expectation value of this Wilson loop is captured by a matrix model. We expect a similar matrix model to be valid also in the case when a surface operator \mathcal{O}_Σ is inserted into the path integral. This suggests how to extrapolate the value of the correlator between \mathcal{O}_Σ and W_{θ_0, ψ_0} to strong coupling by including the strong coupling answer of the matrix model.

In the realization of the surface operator \mathcal{O}_Σ in terms of a probe D3-brane, we calculate the correlator between \mathcal{O}_Σ and W_{θ_0, ψ_0} by evaluating the on-shell action of a string worldsheet that goes between the boundary of $AdS_5 \times S^5$ and which ends on the probe D3-brane in the bulk, and we show that it reproduces the matrix model result.

In the realization of the surface operator \mathcal{O}_Σ in terms of a “bubbling” supergravity solution, we describe the Wilson loop W_{θ_0, ψ_0} by classical worldsheets in the exact “bubbling” supergravity background. Remarkably, the string worldsheet is described by straight line in the auxiliary space of the “bubbling” geometry. The correlator between \mathcal{O}_Σ and W_{θ_0, ψ_0} is then obtained by summing over all string worldsheets that are saddle points of the path integral, reproducing the matrix model result.

Analogous calculations are performed for a supersymmetric 't Hooft loop T_{θ_0, ψ_0} , but in this case there is no known matrix model description of the operator and the gauge theory analysis is performed in the semiclassical regime.

The paper is organized according to the different realizations of a surface operator \mathcal{O}_Σ , and in each realization we perform several calculations. In section 2 we consider the gauge theory description of \mathcal{O}_Σ , where we study conformal anomalies, the vacuum expectation value of \mathcal{O}_Σ , as well as the correlators of \mathcal{O}_Σ with local and Wilson and 't Hooft loop operators W_{θ_0, ψ_0} and T_{θ_0, ψ_0} .

In section 3 we study surface operators and their correlation functions using a probe D3-brane description. Here we evaluate the classical on-shell action of the D3-brane and find the overall scaling anomaly as well as the correlator of \mathcal{O}_Σ with chiral primary operators in $\mathcal{N} = 4$ SYM. We also find the string solution describing the Wilson loop W_{θ_0, ψ_0} in the presence of a surface operator \mathcal{O}_Σ and calculate the correlator between the two operators. We extend this analysis to the case when the Wilson loop is replaced by the 't Hooft loop T_{θ_0, ψ_0} .

In section 4 we turn to the exact supergravity “bubbling” solutions describing surface operators. From the exact supergravity solution we find the correlator of \mathcal{O}_Σ with various local operators \mathcal{O} using holographic renormalization. We then find the exact description of the Wilson loop W_{θ_0, ψ_0} in terms of a classical worldsheet in the full supergravity background corresponding to a surface operator \mathcal{O}_Σ . We show that these worldsheets have a very simple description in the “bubbling” solution and calculate the correlator between \mathcal{O}_Σ and W_{θ_0, ψ_0} by evaluating the on-shell action of the string worldsheet. A similar analysis is considered for the 't Hooft loop T_{θ_0, ψ_0} .

We end with a discussion where, in particular, we speculate about a possible matrix model dual of a surface operator \mathcal{O}_Σ . Some computations are relegated to the appendices.

2. Gauge theory description

In this section we perform calculations with surface operators in $\mathcal{N} = 4$ SYM in the semiclassical approximation of the gauge theory. We start with a brief review of disorder surface operators [1] (see also [7]) and their semiclassical description. We then calculate their expectation value, the correlators of a surface operator with local operators and the correlators of a surface operator with a Wilson and an 't Hooft loop.

2.1 Surface operators in $\mathcal{N} = 4$ SYM

A disorder surface operator \mathcal{O}_Σ supported on a surface $\Sigma \subset R^4$ is characterized by a codimension two singularity for the $\mathcal{N} = 4$ SYM fields.³ In this paper we are interested in maximally supersymmetric surface operators in $\mathcal{N} = 4$ SYM which preserve the maximal subgroup⁴ $\text{PSU}(1, 1|2) \times \text{PSU}(1, 1|2) \times \text{U}(1) \subset \text{PSU}(2, 2|4)$ of the four dimensional superconformal group of $\mathcal{N} = 4$ SYM. The corresponding Euclidean supergroup has an $\text{SO}(1, 3) \times \text{SO}(2) \times \text{SO}(4)$ bosonic subgroup, of which the first component acts on space-time alone, the last is an R -symmetry and the $\text{SO}(2)$ is a diagonal combination of space-time and R -symmetries. $\text{SO}(1, 3)$ is the Euclidean conformal group in two dimensions and the

³For previous work involving codimension two singularities in gauge theory see [16, 17].

⁴This is the supergroup for a maximally supersymmetric surface operator in $R^{1,3}$. In this paper we consider the theory in R^4 , but surface operators exist in both.

only 2-dimensional submanifolds of R^4 which possess this symmetry are R^2 and S^2 , which will be the locus of support of our surface operators.⁵

The symmetries allow a complex scalar field in the $\mathcal{N} = 4$ SYM multiplet to develop a pole near Σ , reducing the $\text{SO}(1, 5) \times \text{SU}(4)$ symmetry of $\mathcal{N} = 4$ SYM to $\text{SO}(1, 3) \times \text{SO}(4) \times \text{U}(1)$.

An alternative way to study a surface operator \mathcal{O}_Σ is to consider the gauge theory on⁶ $AdS_3 \times S^1$ instead of R^4 , the two geometries being related by a Weyl transformation. Then a surface operator \mathcal{O}_Σ is characterized by a non-trivial boundary condition for the fields in the gauge theory near the boundary of $AdS_3 \times S^1$. The advantage of this formulation is that the symmetries of the surface operator \mathcal{O}_Σ are made manifest, as they are realized as isometries of $AdS_3 \times S^1$ instead of as conformal symmetries in R^4 . The choice of surface Σ is now captured by the topology of the $AdS_3 \times S^1$ conformal boundary. For $\Sigma = S^2$ one must consider the AdS_3 metric in global coordinates while for $\Sigma = R^2$ one must consider the metric in Poincaré coordinates.

A surface operator \mathcal{O}_Σ in R^4 induces a non-abelian vortex configuration for the gauge field near the surface operator \mathcal{O}_Σ , which produces a codimension two singularity at Σ . The gauge field configuration breaks the gauge group G to a subgroup $L \subset G$ near Σ . The singularity in the gauge field corresponding to a surface operator \mathcal{O}_Σ with $L = \prod_{l=1}^M \text{U}(N_l) \subset \text{U}(N)$ is given by

$$A = \begin{pmatrix} \alpha_1 \otimes 1_{N_1} & 0 & \cdots & 0 \\ 0 & \alpha_2 \otimes 1_{N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_M \otimes 1_{N_M} \end{pmatrix} d\psi, \quad (2.1)$$

where ψ is the polar angle in the local normal bundle to Σ and 1_n is the n -dimensional unit matrix. The parameters α_i take values in the maximal torus $T^N = R^N/Z^N$ of the gauge group $G = \text{U}(N)$.

The operator \mathcal{O}_Σ also introduces into the path integral two-dimensional θ -angles that couple to the unbroken $\text{U}(1)$'s along Σ . This can be represented by the operator insertion

$$\exp \left(i \sum_{l=1}^M \eta_l \int_\Sigma \text{Tr } F^{(l)} \right), \quad (2.2)$$

where $F^{(l)}$ is the field-strength for each of the unbroken $\text{U}(1)$ factors. The parameters η_i take values in the maximal torus of the S -dual or Langlands dual gauge group ${}^L G$ [1]. Since the Langlands dual of $G = \text{U}(N)$ is also ${}^L G = \text{U}(N)$, we have that the matrix of

⁵The two surfaces are related to each other by a global conformal transformation. On $R^{1,3}$ there are also hyperboloids with geometry H_2 and dS_2 which are conformal to space-like and time-like planes respectively. One should be able to construct maximally supersymmetric surface operators on them too.

⁶Throughout this paper AdS refers always to Euclidean AdS space.

θ -angles of a surface operator \mathcal{O}_Σ is also given by an L -invariant matrix

$$\eta = \begin{pmatrix} \eta_1 \otimes 1_{N_1} & 0 & \cdots & 0 \\ 0 & \eta_2 \otimes 1_{N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_M \otimes 1_{N_M} \end{pmatrix}. \quad (2.3)$$

A maximally supersymmetric surface operator \mathcal{O}_Σ also excites a complex scalar field in $\mathcal{N} = 4$ SYM which we label by $\Phi \equiv \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$. It develops an L -invariant pole near Σ of the form

$$\Phi = \frac{1}{\sqrt{2}z} \begin{pmatrix} \beta_1 + i\gamma_1 \otimes 1_{N_1} & 0 & \cdots & 0 \\ 0 & \beta_2 + i\gamma_2 \otimes 1_{N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_M + i\gamma_M \otimes 1_{N_M} \end{pmatrix}, \quad (2.4)$$

where $z = r e^{i\psi}$ is the complex coordinate in the local normal bundle to Σ . This form of the singularity is completely determined by imposing superconformal invariance and L -invariance.

In summary, a maximally supersymmetric surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM with Levi group $L = \prod_{l=1}^M \mathrm{U}(N_l)$ is labeled by $4M$ L -invariant parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$. The surface operator in R^4 is defined by performing the $\mathcal{N} = 4$ SYM path integral over smooth field fluctuations around the L -invariant singularities (2.1), (2.4) with the insertion of the operator (2.2). We must mod out the path integral by gauge transformations that take values in $L \subset \mathrm{U}(N)$ when restricted to Σ . The surface operator \mathcal{O}_Σ becomes singular whenever the parameters that label the surface operator $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ for $l = 1, \dots, M$ are such that they are invariant under a larger symmetry than L , the group of gauge transformations we have to mod out by when evaluating the path integral.

In the $AdS_3 \times S^1$ picture, the surface operator \mathcal{O}_Σ is described by a non-trivial holonomy of the gauge field (2.1) around the non-contractible S^1 parametrized by the coordinate ψ and by the insertion of (2.2), where Σ is now the conformal boundary of $AdS_3 \times S^1$. The operator \mathcal{O}_Σ is also characterized by a non-singular configuration for the scalar field

$$\Phi = \frac{e^{-i\psi}}{\sqrt{2}} \begin{pmatrix} \beta_1 + i\gamma_1 \otimes 1_{N_1} & 0 & \cdots & 0 \\ 0 & \beta_2 + i\gamma_2 \otimes 1_{N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_M + i\gamma_M \otimes 1_{N_M} \end{pmatrix}, \quad (2.5)$$

where ψ is the coordinate parametrizing the S^1 in $AdS_3 \times S^1$. Therefore, a surface operator \mathcal{O}_Σ is described by a half BPS vacuum state of the gauge theory on $AdS_3 \times S^1$.

Note that the field configurations for $\Sigma = R^2$ and for S^2 are exactly the same when the gauge theory is on $AdS_3 \times S^1$. For $\mathcal{N} = 4$ SYM on R^4 on the other hand, the description of the singularities of the fields produced by a surface operator with $\Sigma = S^2$ of radius a is quite different from $\Sigma = R^2$. One can perform a Weyl transformation from global

$AdS_3 \times S^1$ to R^4 and use that the gauge field and the scalar fields transform as weight zero and weight one fields respectively. Starting with the metric of R^4 in spherical coordinates

$$ds^2 = dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + dx^2, \quad (2.6)$$

and defining

$$\tilde{r}^2 = \frac{(r^2 + x^2 - a^2)^2 + 4a^2 x^2}{4a^2} = \frac{R^2}{(\cosh \rho - \cos \psi)^2}, \quad r = \tilde{r} \sinh \rho, \quad x = \tilde{r} \sin \psi, \quad (2.7)$$

we find the metric

$$ds^2 = \tilde{r}^2 (d\rho^2 + \sinh^2 \rho (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + d\psi^2), \quad (2.8)$$

which is conformal to $AdS_3 \times S^1$ in global coordinates.

In the coordinate system (2.6), a surface operator \mathcal{O}_Σ with $\Sigma = S^2 \subset R^4$ located at $r = a$ produces a singularity on the scalar fields given by (2.4) with the replacement of $z \rightarrow \tilde{r} e^{i\psi}$, where ψ is written in terms of the R^4 coordinates (2.6) using (2.7). The gauge field singularity is exactly the same as before (2.1), where again $d\psi$ is written in terms of the R^4 coordinates (2.6) using (2.7). Finally, the insertion of the θ -angles (2.2) is now obtained by integrating $\text{Tr} F_l$ over the S^2 parametrized by ϑ and φ in (2.6) at $r = a$.

2.2 Surface operator expectation value

We now compute the expectation value of the surface operator \mathcal{O}_Σ in the semiclassical approximation of the gauge theory. We want to calculate the expectation value for both $\Sigma = R^2$ and $\Sigma = S^2$. When the operator is supported on R^2 we anticipate that the expectation value will be one by supersymmetry. For the case $\Sigma = S^2$ the expectation value may depend non-trivially on the radius a of the S^2 .

Since we are dealing with operators defined on a surface Σ , an even dimensional submanifold in spacetime, the surface operator \mathcal{O}_Σ may posses conformal anomalies [10, 11]. The conformal anomalies for surface operators are given by local expressions that depend on the pullback onto the surface Σ of the curvature invariants in spacetime and also on the extrinsic curvature, which encodes the embedding of the surface Σ in spacetime.

The classification of the surface conformal anomalies can be rephrased as a cohomology problem, as in the case of usual conformal anomalies [18]. One must classify solutions to the Wess-Zumino consistency conditions modulo terms that can be obtained as the Weyl variation of a local term. A basis of the Weyl cohomology has been recently presented in [19] (for related previous work on surface anomalies in six dimensions see [12–14]). The surface anomaly depends on three anomaly coefficient c_i , $i = 1, 2, 3$

$$\mathcal{A}_\Sigma = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{h} \left(c_1 R^{(2)} + c_2 g_{mn} \left(h^{\mu\sigma} h^{\nu\rho} - \frac{1}{2} h^{\mu\nu} h^{\rho\sigma} \right) K_{\mu\nu}^m K_{\rho\sigma}^n + c_3 h^{\mu\nu} h^{\rho\sigma} C_{\mu\rho\nu\sigma} \right), \quad (2.9)$$

where $h_{\mu\nu}$ and g_{mn} denote the induced metric on Σ and the spacetime metric respectively while C_{mnpq} and $K_{\mu\nu}^m$ denote the Weyl tensor and the extrinsic curvature, and $C_{\mu\rho\nu\sigma}$ is the

pullback of C_{mnpq} . The anomaly coefficients c_i are the analogs of the well known a and c anomaly coefficients in four dimensions. The expression for c_i depend on the choice of field theory as well as on the choice of surface operator in that field theory.

For the case $\Sigma = R^2 \subset R^4$ we have that $\mathcal{A}_{R^2} = 0$. For $\Sigma = S^2 \subset R^4$ one gets

$$\frac{1}{4\pi} \int_{\Sigma} \sqrt{h} R^{(2)} = 2, \quad g_{mn} \left(h^{\mu\sigma} h^{\nu\rho} - \frac{1}{2} h^{\mu\nu} h^{\rho\sigma} \right) K_{\mu\nu}^m K_{\rho\sigma}^n = C_{\mu\rho\nu\sigma} = 0. \quad (2.10)$$

Therefore $\mathcal{A}_{S^2} = 2c_1$.

Under a conformal transformation $g_{mn} \rightarrow e^{2\sigma} g_{mn}$ the expectation value of a surface operator \mathcal{O}_{Σ} supported on $S^2 \subset R^4$ changes by an amount proportional to the anomaly

$$\delta_{\sigma} \langle \mathcal{O}_{\Sigma} \rangle = 2c_1 \langle \mathcal{O}_{\Sigma} \rangle. \quad (2.11)$$

Therefore the expectation value of \mathcal{O}_{Σ} is determined in terms of the radius a of the S^2 by

$$\langle \mathcal{O}_{\Sigma} \rangle \propto a^{2c_1}. \quad (2.12)$$

The precise form of the expectation value of \mathcal{O}_{Σ} now depends on the chosen operator and on the theory under discussion, which uniquely determine the anomaly coefficients c_i .

We now focus on our specific case of the maximally supersymmetric surface operators \mathcal{O}_{Σ} in $\mathcal{N} = 4$ SYM and calculate their vacuum expectation value in the semiclassical approximation. This is achieved by evaluating the classical SYM action on the field configuration produced by the operator \mathcal{O}_{Σ}

$$\langle \mathcal{O}_{\Sigma} \rangle = \exp(-\mathcal{S})|_{\text{surface}}. \quad (2.13)$$

It is easiest to perform this calculation by considering the field configuration produced by a surface operator \mathcal{O}_{Σ} on $AdS_3 \times S^1$. The relevant part of the action of $\mathcal{N} = 4$ SYM on $AdS_3 \times S^1$ is given by

$$\mathcal{L} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(|D\Phi|^2 + \frac{R^{(4)}}{6} |\Phi|^2 \right), \quad (2.14)$$

where $\frac{R^{(4)}}{6} |\Phi|^2$ is the conformal coupling for the scalars and $R^{(4)} = -6$ is the scalar curvature of $AdS_3 \times S^1$. The surface operator \mathcal{O}_{Σ} induces a non-trivial field configuration on the field Φ (2.5) on $AdS_3 \times S^1$ which depends only on the angular coordinate ψ of the S^1 through $\Phi = \Phi_0 e^{-i\psi}$, where Φ_0 is a constant diagonal matrix (2.5). The operator \mathcal{O}_{Σ} also induces a nontrivial gauge field configuration (2.1) which nevertheless commutes with Φ_0 , so that $D\Phi = \partial\Phi$. Therefore $|D\Phi|^2 = |\Phi|^2$ and the two terms in the Lagrangian (2.14) exactly cancel each, so that

$$\langle \mathcal{O}_{\Sigma} \rangle = 1. \quad (2.15)$$

This result holds regardless of whether we use the Poincaré patch or global AdS_3 metric in the field theory, so that $\langle \mathcal{O}_{\Sigma} \rangle = 1$ for both $\Sigma = R^2$ and $\Sigma = S^2$.

The same result can also be derived by studying $\mathcal{N} = 4$ SYM on R^4 . In this case the scalars do not have a mass term. However, one must include in the action a boundary term

along the surface operator \mathcal{O}_Σ that guaranties that the boundary terms in the equations of motion vanish. To see that, consider the relevant part of the action in flat space

$$\mathcal{S} = \frac{1}{g_{\text{YM}}^2} \int d^4x \sqrt{g} g^{mn} \text{Tr} (D_m \Phi D_n \bar{\Phi}) \quad (2.16)$$

with the metric (2.8). In deriving the equations of motion there is a boundary term near the locus of the surface operator at a cutoff $\rho = \rho_0$

$$\delta \mathcal{S} = \frac{1}{g_{\text{YM}}^2} \int d\vartheta d\varphi d\psi \tilde{r}^2 \sinh^2 \rho_0 \sin \vartheta (\delta \Phi \partial_\rho \bar{\Phi} + \delta \bar{\Phi} \partial_\rho \Phi) . \quad (2.17)$$

This does not vanish, since Φ diverges. To fix this we need to add to the action the boundary term

$$\mathcal{S}_{\text{boundary}} = -\frac{1}{g_{\text{YM}}^2} \int d\vartheta d\varphi d\psi \frac{1}{a} \tilde{r}^3 \sinh^3 \rho_0 \sin \vartheta \text{Tr} (\Phi \bar{\Phi}) . \quad (2.18)$$

The variation of the bulk and boundary action would now give the boundary equation $\partial_\rho \Phi - \tilde{r} \sinh \rho_0 \Phi / a = 0$, which is indeed satisfied by our solution.

Evaluating the classical action with a cutoff ρ_0 gives the divergent result

$$\mathcal{S} = \frac{1}{g_{\text{YM}}^2} \int d\rho d\psi d\vartheta d\varphi \frac{\cosh \rho \sinh^2 \rho \sin \vartheta}{\cosh \rho - \cos \psi} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2) = 4\pi^2 \sinh^2 \rho_0 \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2) . \quad (2.19)$$

The boundary term (2.18), though, exactly cancels this bulk piece leading to a total action that vanishes, exactly as we found in the $AdS_3 \times S^1$ picture.

It follows from (2.12) and from $\langle \mathcal{O}_\Sigma \rangle = 1$ for a spherical surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM that the anomaly coefficient c_1 for this theory (and this family of surface operators) is $c_1 = 0$. It would be interesting to calculate separately the anomaly coefficients c_2, c_3 , which can be done by computing the anomaly for more general surfaces Σ as well as by considering $\mathcal{N} = 4$ SYM in more general curved four dimensional backgrounds.

2.3 Correlator of surface operators with local operators

We now proceed to calculate the correlation function of a surface operator \mathcal{O}_Σ with various local operators in $\mathcal{N} = 4$ SYM in R^4 . We consider the correlator with the dimension $\Delta = 2, 3$ chiral primary operators (CPO's) \mathcal{O}_Δ of $\mathcal{N} = 4$ SYM, the traceless stress-energy tensor T_{mn} and with the U(1) spacetime symmetry current J_m^ψ which generates rotations in the space transverse to the surface Σ .

In the semiclassical approximation, the correlation function of a surface operator \mathcal{O}_Σ with a local operator \mathcal{O} is obtained by evaluating \mathcal{O} in the background field produced by the surface operator \mathcal{O}_Σ

$$\frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O} \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \mathcal{O}|_{\text{surface}} . \quad (2.20)$$

If instead we consider the surface operator \mathcal{O}_Σ as a non-trivial boundary condition for $\mathcal{N} = 4$ SYM on $AdS_3 \times S^1$, the correlation function of \mathcal{O}_Σ with a local operator \mathcal{O} in R^4

corresponds to the vacuum expectation value of the local operator \mathcal{O} in the state $|\mathcal{O}_\Sigma\rangle$. Thus on $AdS_3 \times S^1$ the correlation function (2.20) in R^4 corresponds to

$$\langle \mathcal{O}^I \rangle_{\mathcal{O}_\Sigma}. \quad (2.21)$$

In all these computations the position dependence of the local operator is determined by conformal Ward-Takahashi identities. In the case of $\Sigma = R^2 \subset R^4$ the correlator of a surface operator \mathcal{O}_Σ with a local operator \mathcal{O} of dimension Δ scales with the distance r between the local operator \mathcal{O} and the plane where the surface operator \mathcal{O}_Σ is supported by⁷

$$\frac{\langle \mathcal{O} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \frac{C_{\mathcal{O}}}{r^\Delta}. \quad (2.22)$$

The quantity that we need to compute for the various local operators \mathcal{O} is $C_{\mathcal{O}}$.

When the surface operator \mathcal{O}_Σ is supported on $S^2 \subset R^4$, the correlator is given by

$$\frac{\langle \mathcal{O} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \frac{C_{\mathcal{O}}}{\tilde{r}^{2\Delta}}. \quad (2.23)$$

where \tilde{r} is the conformally invariant distance (2.7)

$$\tilde{r} = \frac{\sqrt{(r^2 + x^2 - a^2)^2 + 4a^2x^2}}{2a}, \quad (2.24)$$

combining a , the radius of the S^2 , the radial position of the local operator in the R^3 including the surface, r and the transverse distance to the local operator, x . In principle the proportionality constant in (2.22), (2.23) $C_{\mathcal{O}}$ should be the same for $\Sigma = S^2$ and $\Sigma = R^2$, except that there is the possibility that some contributions that exist for the sphere are “pushed to infinity” in the case of the plane and will not show up. This does not happen in the specific calculations we performed but may show up once one includes quantum corrections.

When one considers the gauge theory on $AdS_3 \times S^1$ the surface operator \mathcal{O}_Σ is supported at the boundary of AdS_3 . Therefore, there is no invariant distance like r or \tilde{r} in R^4 . In this case

$$\langle \mathcal{O} \rangle_{\mathcal{O}_\Sigma} = C_{\mathcal{O}}. \quad (2.25)$$

The CPO's are scalar operators transforming in the $(0, \Delta, 0)$ representation of the $SU(4)$ R-symmetry. The operators are given by

$$\mathcal{O}_\Delta^I = \frac{(8\pi^2)^{\Delta/2}}{\lambda^{\Delta/2} \sqrt{\Delta}} C_{i_1 \dots i_\Delta}^I \text{Tr}(\phi^{i_1} \dots \phi^{i_\Delta}), \quad (2.26)$$

where $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling and $Y^I = C_{i_1 \dots i_\Delta}^I x^{i_1} \dots x^{i_\Delta}$ are the $SO(6)$ scalar spherical harmonics. The operators (2.26) are normalized such that their two point function is unit normalized

$$\langle \mathcal{O}_\Delta^I(x) \mathcal{O}_\Delta^J(y) \rangle = \frac{\delta^{IJ}}{|x - y|^{2\Delta}}. \quad (2.27)$$

⁷For operators that are not scalars, the tensor structure of the correlator is determined by Lorentz invariance.

A surface operator \mathcal{O}_Σ preserves an $\text{SO}(4)$ subgroup of the $\text{SU}(4)$ R-symmetry group. Therefore, the correlation function of a chiral primary operator \mathcal{O}_Δ^I with a surface operator \mathcal{O}_Σ is nontrivial only if the CPO is an $\text{SO}(4)$ singlet. By looking at the decomposition of the $(0, \Delta, 0)$ representation of $\text{SO}(6) \simeq \text{SU}(4)$ under the obvious $\text{SO}(4) \times \text{SO}(2)$ subgroup one finds that there are $\Delta + 1$ CPO's of dimension Δ which are singlets under $\text{SO}(4)$. These $\text{SO}(4)$ invariant operators have charges $k = -\Delta, -\Delta + 2, \dots, \Delta - 2, \Delta$ under $\text{SO}(2)$. We label the $\text{SO}(4)$ invariant spherical harmonics by $Y^{\Delta, k} = C_{i_1 \dots i_\Delta}^{\Delta, k} x^{i_1} \dots x^{i_\Delta}$ (for more details see appendix A). With these spherical harmonics we can write down the $\text{SO}(4)$ invariant CPO's as

$$\mathcal{O}_{\Delta, k} = \frac{(8\pi^2)^{\Delta/2}}{\lambda^{\Delta/2} \sqrt{\Delta}} C_{i_1 \dots i_\Delta}^{\Delta, k} \text{Tr}(\phi^{i_1} \dots \phi^{i_\Delta}). \quad (2.28)$$

Using the explicit expression for the $\text{SO}(4)$ invariant spherical harmonics with $\Delta = 2, 3$, we can write down the unit normalized CPO's

$$\begin{aligned} \mathcal{O}_{2,0} &= \frac{4\pi^2}{\sqrt{6}\lambda} \text{Tr} \left(4\Phi \bar{\Phi} - \sum_{I=1}^4 \phi^I \phi^I \right); & \mathcal{O}_{2,2} &= \frac{8\pi^2}{\sqrt{2}\lambda} \text{Tr}(\Phi^2); \\ \mathcal{O}_{3,1} &= \frac{8\pi^3}{\lambda^{3/2}} \text{Tr} \left(2\Phi^2 \bar{\Phi} - \Phi \sum_{I=1}^4 \phi^I \phi^I \right); & \mathcal{O}_{3,3} &= \frac{32\pi^3}{\sqrt{6}\lambda^{3/2}} \text{Tr}(\Phi^3), \end{aligned} \quad (2.29)$$

where as before $\Phi = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$ and $\mathcal{O}_{\Delta, -k} = \bar{\mathcal{O}}_{\Delta, k}$.

The semiclassical correlations function (2.20) of these CPO's with the surface operator \mathcal{O}_Σ are then given by

$$\begin{aligned} \frac{\langle \mathcal{O}_{2,0} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{|z|^2} \frac{8\pi^2}{\sqrt{6}\lambda} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2); & \frac{\langle \mathcal{O}_{2,2} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{z^2} \frac{4\pi^2}{\sqrt{2}\lambda} \sum_{l=1}^M N_l (\beta_l + i\gamma_l)^2 \\ \frac{\langle \mathcal{O}_{3,1} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{z|z|^2} \frac{8\pi^3}{\sqrt{2}\lambda^{3/2}} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2)(\beta_l + i\gamma_l); \\ \frac{\langle \mathcal{O}_{3,3} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{z^3} \frac{8\pi^3}{\sqrt{3}\lambda^{3/2}} \sum_{l=1}^M N_l (\beta_l + i\gamma_l)^3. \end{aligned} \quad (2.30)$$

The remaining correlators can be obtained by complex conjugating (2.30).

These result are purely classical and they may receive quantum corrections. The surface operator \mathcal{O}_Σ breaks near Σ the gauge group from $\text{U}(N)$ to $L = \prod_{l=1}^M \text{U}(N_l)$, and also breaks R-symmetry group down to $\text{SO}(4)$. This breaking may introduce different interactions among the different fields at the quantum level. We therefore expect the quantum corrections to depend on the choice of Levi group given by the rank of the unbroken gauge groups N_1, \dots, N_M . The calculations performed using probe D3-branes and the “bubbling” supergravity solution in the next two sections indicate, however, that the loop corrections terminate after a finite number of loops! The perturbative corrections to the correlator of \mathcal{O}_Σ with $\mathcal{O}_{\Delta, k}$ terminate at order⁸ $\lambda^{(\Delta - |k|)/2}$. It would be interesting to understand the truncation of the loop corrections directly in the gauge theory.

⁸Note that $\Delta - |k|$ is always even, so we get a standard perturbation expansion in the 't Hooft coupling λ .

We now consider the correlator of \mathcal{O}_Σ with the stress-energy tensor T_{mn} of $\mathcal{N} = 4$ SYM in R^4 , which was already computed in [7]. In the presence of a planar surface operator in R^4 the correlator is fixed up to function h

$$\frac{\langle T_{\mu\nu} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = h \frac{\eta_{\mu\nu}}{r^4}, \quad \frac{\langle T_{ij} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \frac{h}{r^4} [4n_i n_j - 3\delta_{ij}], \quad \langle T_{\mu i} \cdot \mathcal{O}_\Sigma \rangle = 0. \quad (2.31)$$

Here $x^m = (x^\mu, x^i)$, where x^μ are coordinates along $\Sigma = R^2$ and $n^i = x^i/r$ is the unit normal vector to the planar surface. If $\mathcal{N} = 4$ SYM is on $AdS_3 \times S^1$ the expectation value of the stress-energy tensor T_{mn} in the presence of a surface operator \mathcal{O}_Σ is given by

$$\langle T_{ab} \rangle_{\mathcal{O}_\Sigma} = h g_{ab} \quad \langle T_{\psi\psi} \rangle_{\mathcal{O}_\Sigma} = -3h, \quad (2.32)$$

where g_{ab} is the metric on AdS_3 and ψ is the coordinate on the S^1 .

The piece of the stress-energy tensor of $\mathcal{N} = 4$ SYM which involves the gauge field and the complex scalar field Φ excited by the surface operator \mathcal{O}_Σ is given by (see e.g. [20])⁹

$$T_{mn} = \frac{2}{3g_{\text{YM}}^2} \text{Tr} \left[2D_m \Phi D_n \bar{\Phi} + 2D_m \bar{\Phi} D_n \Phi - g_{mn} |D\Phi|^2 - \Phi D_m D_n \bar{\Phi} - \bar{\Phi} D_m D_n \Phi \right. \\ \left. + \frac{1}{4} g_{mn} \Phi D^2 \bar{\Phi} + \frac{1}{4} g_{mn} \bar{\Phi} D^2 \Phi \right] + \frac{2}{g_{\text{YM}}^2} \text{Tr} \left[-F_{ml} F_{nl} + \frac{1}{4} g_{mn} F_{lp} F_{lp} \right]. \quad (2.33)$$

Using the semiclassical formula (2.20) we find that

$$h = -\frac{2}{3g_{\text{YM}}^2} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2). \quad (2.34)$$

or equivalently

$$\langle T_{ab} \rangle_{\mathcal{O}_\Sigma} = -\frac{2}{3g_{\text{YM}}^2} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2) g_{ab} \\ \langle T_{\psi\psi} \rangle_{\mathcal{O}_\Sigma} = \frac{2}{g_{\text{YM}}^2} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2). \quad (2.35)$$

In this case, we expect that the only perturbative correction appears at the one loop level, of order λ , which as we will see is captured by the “bubbling” supergravity solution.

In [21] the analogous coefficient h for Wilson and ’t Hooft loop operators was dubbed the scaling weight, the name suggesting that it should generalize the notion of conformal dimension of local conformal fields to extended objects. The analogy with conformal dimension of a local operators is not complete. The conformal dimension of a local operator measures its behavior under conformal transformations, while for surface operators there are three different anomalies under such transformations which we dubbed c_i in (2.9). We expect the scaling weight h to be related to the anomaly coefficients c_i and it would be interesting to find the explicit relation.

⁹We neglect the scalar potential term, as it does not contribute to the correlator.

We also calculate the correlator of \mathcal{O}_Σ with the conserved current J_m^ψ . This current generates the U(1) symmetry that acts by shifts on the S^1 of the $AdS_3 \times S^1$ geometry on which the gauge theory is defined. However, it is not a symmetry of the surface operator \mathcal{O}_Σ as the singularity induced on the scalar field (2.4) is not U(1) invariant. In conformal field theories, the current associated with a spacetime symmetry is constructed by contracting the stress-energy tensor T_{mn} with the conformal Killing vector ξ_G generating the symmetry as $j_m^G = T_{mn}\xi_G^n$. Therefore, the correlator of J_m^ψ with \mathcal{O}_Σ is given by

$$\langle J_\psi^\psi \rangle_{\mathcal{O}_\Sigma} = \langle T_{\psi\psi} \rangle_{\mathcal{O}_\Sigma} = \frac{2}{g_{\text{YM}}^2} \sum_{l=1}^M N_l (\beta_l^2 + \gamma_l^2). \quad (2.36)$$

2.4 Correlator with Wilson and 't Hooft loop operator

The last calculation we perform is the correlator of a surface operator \mathcal{O}_Σ with a Wilson loop operator and also make comments about the correlator of a surface operator \mathcal{O}_Σ with an 't Hooft loop operator. This combination of operators is particularly interesting since in four dimensional space a two dimensional surface can be linked by a curve.

We take the Wilson loop supported on a curve C to link the surface Σ on which the surface operator \mathcal{O}_Σ is supported. The simplest way to realize this configuration is by taking the surface $\Sigma = R^2 \subset R^4$ and consider a circular loop $C = S^1$ in the orthogonal plane. A natural choice of Wilson loop to consider is the 1/4 BPS circular Wilson loop with periodic coupling to the scalar fields with arbitrary phase shift ψ_0 [22]

$$W_{\psi_0} = \frac{1}{N} \text{Tr} \exp \int (iA_\psi + r \cos(\psi - \psi_0) \phi^5 - r \sin(\psi - \psi_0) \phi^6) d\psi. \quad (2.37)$$

Here ϕ^5 and ϕ^6 are the real and imaginary part of the complex field turned on by the surface operator \mathcal{O}_Σ , given by $\Phi \equiv \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$.

We consider an even more general Wilson loop operator, presented in [15] and studied further in [23], with an extra coupling to a third real scalar field ϕ^1 in the $\mathcal{N} = 4$ SYM multiplet, with a relative coupling given by the angle θ_0

$$W_{\theta_0, \psi_0} = \frac{1}{N} \text{Tr} \exp \int d\psi \left(iA_\psi + |z| \cos \theta_0 \phi^1 + \sqrt{2} \sin \theta_0 \Re(z\Phi e^{-i\psi_0}) \right). \quad (2.38)$$

For $\theta_0 = \pi/2$ it reduces to (2.37) (note that the ψ dependence comes now from $z = re^{i\psi}$), while for $\theta_0 = 0$ it's the well studied maximally supersymmetric circular loop [24, 25].

Note that this Wilson loop operator depends on two parameters θ_0 and ψ_0 and operators with different values of these parameters are quite different. In particular, the supercharges that are preserved by the Wilson loop W_{θ_0, ψ_0} depend on these parameters, though they are always compatible with those of the surface operator. This is proven in appendix B.

Evaluating the correlator of a surface operator \mathcal{O}_Σ with the Wilson loop (2.38) in the semiclassical approximation amounts to evaluating the Wilson loop operator (2.38) on the fields produced by the surface operator (2.1), (2.4). The result is given by

$$\frac{\langle W_{\theta_0, \psi_0} \cdot \mathcal{O}_\Sigma \rangle_{\text{class}}}{\langle \mathcal{O}_\Sigma \rangle} = \sum_{l=1}^M \frac{N_l}{N} \exp \left[2\pi \sin \theta_0 (\beta_l \cos \psi_0 + \gamma_l \sin \psi_0) + 2\pi i \alpha_l \right]. \quad (2.39)$$

The extra scalar ϕ^1 does not contribute at the classical level, as the surface operator \mathcal{O}_Σ does not excite it.

In the absence of a surface operator \mathcal{O}_Σ , it is conjectured in [23] that the exact expectation value of the Wilson loop W_{θ_0, ψ_0} is given by summing up all ladder diagrams in the Feynman gauge, which is captured by a matrix model. In the limit of large 't Hooft coupling λ it has an asymptotic expansion with two saddle points [23] which can be written schematically as

$$\langle W_{\theta_0, \psi_0} \rangle \simeq \exp \left[\sqrt{\lambda} \cos \theta_0 \right] + \exp \left[-\sqrt{\lambda} \cos \theta_0 \right]. \quad (2.40)$$

This result, including both saddle points can be reproduced from string theory.

The expression (2.40) can be regarded as coming purely from the scalar field ϕ^1 , which appears with a factor of $\cos \theta_0$ in the loop (2.38), while the other two scalars are present to guarantee supersymmetry.

The presence of the surface operator can, of course, modify the contribution of ϕ^1 to the expectation value of the Wilson loop (2.38), but we now argue that its contribution is well under control. Indeed this is what happens for the Wilson loop W_{θ_0, ψ_0} in the presence of a chiral primary local operator [26] (see also [27, 28]), where the exact result is given by a normal matrix model, but the asymptotics are still governed by an exponent like in (2.40).

In particular, one possible effect of the surface operator \mathcal{O}_Σ on the scalar ϕ^1 in the Wilson loop (2.38) is in breaking of the $U(N)$ gauge group to the Levi group $L = \prod_{l=1}^M U(N_l)$, thus reducing the number of degrees of freedom in the system. Now, in each $U(N_l)$ sector, ϕ^1 gives the same matrix model result with the replacement $\lambda \rightarrow \lambda N_l / N$. Assuming that all the N_l are still large, so that we may still use the approximation in (2.40), we predict that the correlator of the Wilson loop W_{θ_0, ψ_0} with a surface operator \mathcal{O}_Σ at strong coupling is given by

$$\frac{\langle W_{\theta_0, \psi_0} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \sum_{l=1}^M \frac{N_l}{N} \left(\exp \left[\sqrt{\frac{\lambda N_l}{N}} \cos \theta_0 \right] + \exp \left[-\sqrt{\frac{\lambda N_l}{N}} \cos \theta_0 \right] \right) \times \\ \times \exp \left[2\pi \sin \theta_0 (\beta_l \cos \psi_0 + \gamma_l \sin \psi_0) + 2\pi i \alpha_l \right]. \quad (2.41)$$

We will later see that this result is reproduced by performing a string theory calculation in the “bubbling” supergravity background dual to a surface operator.

We note that unlike local operators, this class of Wilson loops are sensitive to the Aharonov-Bohm phase (2.1) induced by the surface operator \mathcal{O}_Σ .

Similarly to the Wilson loop, one can consider an 't Hooft loop, which should yield the S-dual result (accounting of course also for the S-duality transformation of the surface operator parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$). Unlike the Wilson loop, the 't Hooft loop does not have a simple weak-coupling description. Rather, much like the surface operator \mathcal{O}_Σ , it can be thought of as requiring certain singularities for the gauge field and scalar field along the loop.

The 't Hooft loop T_{θ_0, ψ_0} we want to consider should be the dual of the Wilson loop (2.38), which couples to three scalars and depends on the parameters θ_0 and ψ_0 . An 't Hooft loop can be represented by a Dirac monopole embedded along some $U(1)$

subgroup of $U(N)$ with the accompanying scalar source. The field-strength around the monopole has the form

$$F = \frac{1}{2} T^0 \sin \theta d\theta d\phi, \quad (2.42)$$

where θ and ϕ are local coordinates on a sphere surrounding the loop, and T^0 is a generator in the Cartan subalgebra of $U(N)$. This should be the behavior of the gauge field close to the 't Hooft loop T_{θ_0, ψ_0} for the straight line. In the classical approximation this is the exact solution. So consider now the spherical surface operator \mathcal{O}_Σ wrapping the straight 't Hooft loop T_{θ_0, ψ_0} . The surface operator couples to the magnetic field produced by the 't Hooft operator through the parameters η in (2.2) entering the definition of \mathcal{O}_Σ

$$\exp \left(i\eta \int_\Sigma \text{Tr} \frac{T^0}{2} \right) = \exp (2\pi i\eta). \quad (2.43)$$

This produces the S -dual of the phase $\exp(2\pi i\alpha)$ in (2.41) obtained from the correlator of \mathcal{O}_Σ with the Wilson loop (2.38), as $\alpha \rightarrow \eta$ under S -duality [1].

The contribution from the scalars Φ and $\bar{\Phi}$ is almost identical to the one in the Wilson loop calculation, as the effect of the singularity on the scalar fields produced by the 't Hooft loop can be modeled in the semiclassical approximation by inserting into the $\mathcal{N} = 4$ SYM path integral the operator

$$\exp \left(\frac{4\pi}{g_{\text{YM}}^2} \int \text{Tr} \left(T^0 \Re(z\Phi e^{-i\psi_0}) \right) d\psi \right). \quad (2.44)$$

Inserting the field (2.4) produced by the surface operator \mathcal{O}_Σ in (2.44) gives the S -dual of the Wilson loop result (2.39) as $|\beta + i\gamma| \rightarrow 4\pi/g_{\text{YM}}^2 |\beta + i\gamma|$ under S -duality [1]. The contribution of ϕ^1 as in (2.40) has so far not been reproduced from the 't Hooft operator in a gauge-theory calculation even in the absence of the surface operator \mathcal{O}_Σ , and we do not attempt to recover that here. We will see in section 3 and in section 4, where we calculate the Wilson loop and 't Hooft loop using string theory, that the resulting expressions are indeed related to each-other by S -duality.

We note that the correlator of \mathcal{O}_Σ with an 't Hooft operator is sensitive to the two dimensional θ -angles (2.2) that appear in the definition of a surface operator \mathcal{O}_Σ .

3. Probe description in $AdS_5 \times S^5$

In this section we perform calculations with surface operators in $\mathcal{N} = 4$ SYM in the probe approximation of the dual Type IIB string theory on $AdS_5 \times S^5$. We start with a description of surface operators in $\mathcal{N} = 4$ SYM in terms of D3-branes in $AdS_5 \times S^5$. We then calculate their expectation value, the correlators of a surface operator with CPO's and the correlators of a surface operator with a Wilson and an 't Hooft loop.

3.1 Surface operator as D-branes in $AdS_5 \times S^5$

A surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM corresponds in the probe picture to having a D3-brane in $AdS_5 \times S^5$ ending on the boundary along the prescribed surface Σ . This D3-brane solution in $AdS_5 \times S^5$ was first studied by [29].

We start by describing the details of the $AdS_3 \times S^1$ D3-brane solution that we need to perform the various calculations in this section. We take the metric on the S^5 to be

$$d\Omega_5 = \cos^2 \theta d\Omega_3 + d\theta^2 + \sin^2 \theta d\phi^2, \quad (3.1)$$

which makes manifest the $SO(4) \times SO(2) \subset SO(6)$ symmetry of the S^5 . For the surface operator \mathcal{O}_Σ supported on $\Sigma = R^2$ it is convenient to write the bulk AdS metric in two different slicings

$$ds_{(1)}^2 = \frac{1}{y^2} (dy^2 + dx_0^2 + dx_1^2 + dr^2 + r^2 d\psi^2), \quad (3.2)$$

$$ds_{(2)}^2 = \frac{\cosh^2 u}{z^2} (dz^2 + dx_0^2 + dx_1^2) + du^2 + \sinh^2 u d\psi^2, \quad (3.3)$$

where we work in units where the AdS_5 and S^5 radius of curvature is one so that $L^4 = 4\pi g_s N l_s^4 = 1$. The two coordinate systems are related by

$$y = \frac{z}{\cosh u}, \quad r = z \tanh u. \quad (3.4)$$

The first choice of coordinates (3.2) is more suitable when considering the dual $\mathcal{N} = 4$ SYM theory on R^4 . The second one (3.3) is more suitable when $\mathcal{N} = 4$ SYM is on $AdS_3 \times S^1$, where $AdS_3 \times S^1$ is the metric on the conformal boundary located at $u \rightarrow \infty$.

We choose the RR four-form potential in the analog of the Fefferman-Graham gauge [30] for the metric, so that the four-form has no components transverse to the AdS_5 boundary. Therefore, we take the RR four-form potential in the metric (3.2) and (3.3) to be given by

$$C_4^{(1)} = \frac{r}{y^4} dx_0 \wedge dx_1 \wedge dr \wedge d\psi, \quad (3.5)$$

$$C_4^{(2)} = \frac{\cosh^4 u}{z^3} dz \wedge dx_0 \wedge dx_1 \wedge d\psi. \quad (3.6)$$

The worldvolume coordinates of the D3-brane embedding in the coordinate system (3.2) are given by x_0, x_1, r and ψ , and the D3-brane has non-trivial embedding functions $y = y(r)$ and on the S^5 in the coordinate system (3.1): $\theta = \pi/2$ and $\phi = \phi(\psi)$. With this ansatz the D3-brane action is given by

$$\mathcal{S}_{D3} = T_{D3} \int dx_0 dx_1 dr d\psi \frac{1}{y^4} \sqrt{(1 + y'^2)(r^2 + y^2 \dot{\phi}^2)} - T_{D3} \int dx_0 dx_1 dr d\psi \frac{r}{y^4} \quad (3.7)$$

where the D3-brane tension is given by $T_{D3} = \frac{N}{2\pi^2}$ (in the units where $L^4 = 1$). The half-BPS solution is given by [29]

$$y(r) = \frac{1}{\kappa} r, \quad \psi + \phi = \phi_0, \quad (3.8)$$

where κ and ϕ_0 are integration constants.

In the coordinate system (3.3) the D3-brane solution has worldvolume coordinates z, x^0, x^1 and ψ and the embedding is given by $\theta = \pi/2$ in (3.1) and

$$\sinh u_0 = \kappa, \quad \psi + \phi = \phi_0. \quad (3.9)$$

The induced metric on the brane is that of $AdS_3 \times S^1$, so that we have the freedom of turning on a Wilson line for the gauge field A and dual gauge field \tilde{A} living on the D3-brane worldvolume along the non-contractible S^1 . Therefore, the D3-brane solution ending on $\Sigma = R^2$ on the boundary depends on four parameters [1], which can be identified with those of the gauge theory description of a surface operator \mathcal{O}_Σ via¹⁰

$$\begin{aligned}\alpha &= \oint \frac{A}{2\pi}, \\ \beta + i\gamma &= \frac{\sqrt{\lambda}}{2\pi} \sinh u_0 e^{i\phi_0}, \\ \eta &= \oint \frac{\tilde{A}}{2\pi}.\end{aligned}\tag{3.10}$$

Therefore, in the probe approximation a surface operator \mathcal{O}_Σ with unbroken gauge group $L = \prod_{l=1}^M U(N_L)$ and labeled by the $4M$ parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ corresponds to a configuration of M stacks of D3 branes in $AdS_5 \times S^5$ ending on the boundary on the surface Σ , with each stack consisting of N_l coincident D3 branes. In appendix C we study the supersymmetry preserved by the probe D3-brane brane and show that it is the same as the supersymmetry preserved by the surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM.

For the probe D3-brane description of the surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM supported on $\Sigma = S^2$ it is convenient to write the bulk AdS_5 metric in two different slicings

$$ds_{(3)}^2 = \frac{1}{y^2} (dy^2 + dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + dx^2), \tag{3.11}$$

$$ds_{(4)}^2 = \cosh^2 u (d\rho^2 + \sinh^2 \rho (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)) + du^2 + \sinh^2 u d\psi^2. \tag{3.12}$$

The two coordinate systems are related by

$$y = \frac{a}{\cosh u \cosh \rho - \sinh u \cos \psi}, \quad r = y \cosh u \sinh \rho, \quad x = y \sinh u \sin \psi, \tag{3.13}$$

where a is the radius of the $\Sigma = S^2$ on which the dual surface operator \mathcal{O}_Σ is supported. Again, the choice of coordinates (3.11) is appropriate when considering the gauge theory on R^4 while the one in (3.12) is appropriate when the gauge theory is on $AdS_3 \times S^1$.

The associated expressions for the RR four-form potential are given by:

$$C_4^{(3)} = \frac{r^2 \sin \vartheta}{y^4} dr \wedge d\vartheta \wedge d\varphi \wedge dx, \tag{3.14}$$

$$C_4^{(4)} = \cosh^4 u \sinh^2 \rho \sin \psi d\rho \wedge d\vartheta \wedge d\varphi \wedge d\psi. \tag{3.15}$$

The D3-brane worldvolume coordinates in the metric (3.12) are ρ, ϑ, φ and ψ , and the embedding is again given by $\theta = \pi/2$ in (3.1) and:

$$\sinh u_0 = \kappa, \quad \psi + \phi = \phi_0. \tag{3.16}$$

¹⁰Note that in the units $L^4 = 1$, we can write alternatively $\beta + i\gamma = \frac{1}{2\pi l_s^2} \sinh u_0 e^{i\phi_0}$ (see also equation (4.9)).

In the coordinate system (3.11) the D3-brane embedding translates into

$$\frac{(r^2 + x^2 + y^2 - a^2)^2 + 4a^2x^2}{4a^2y^2} = \kappa^2. \quad (3.17)$$

The map between the parameters of the D3-brane solution and the parameters defining the dual gauge theory surface operator \mathcal{O}_Σ is also given by (3.10). As explained earlier, considering M stacks of D3 branes yields a surface operator \mathcal{O}_Σ with parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ for $l = 1, \dots, M$.

3.2 Expectation value in the probe approximation

The expectation value of the surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM corresponding to a single probe D3-brane is given in the leading $N \gg 1$ and $\lambda \gg 1$ approximation by

$$\langle \mathcal{O}_\Sigma \rangle = e^{-\mathcal{S}_{D3}|_{\text{on-shell}}}. \quad (3.18)$$

This corresponds to a surface operator \mathcal{O}_Σ with unbroken gauge group $L = \text{U}(1) \times \text{SU}(N-1)$ and with parameters $(\alpha, \beta, \gamma, \eta)$. Therefore, we have to evaluate the classical action of the D3-brane solutions in the previous subsection.

For the case of a surface operator \mathcal{O}_Σ on $\Sigma = R^2$, we find that the action of the dual D3-brane in the metric (3.2) is given by

$$\mathcal{S}_{D3} = \frac{N}{2\pi^2} \int dx_0 dx_1 dr d\psi \frac{1}{y^4} \left[\sqrt{(1+y'^2)(r^2 + y^2\dot{\phi}^2)} - r \right], \quad (3.19)$$

where $\mathcal{S}_{D3} = \mathcal{S}_{\text{DBI}} - \mathcal{S}_{\text{WZ}}$. In order to evaluate the classical action it is not enough to plug in the solution $\dot{\phi} = -1$ and $y = r/\kappa$ into the action. Examining the variation of the action one finds a boundary term that needs to be cancelled. That is due to the fact that y' is a non-zero constant and to get a good variational problem it is necessary to subtract

$$y' \frac{\partial \mathcal{L}}{\partial y'} = \frac{N}{2\pi^2} \frac{y'^2}{y^4} \sqrt{\frac{r^2 + y^2\dot{\phi}^2}{1 + y'^2}}. \quad (3.20)$$

Therefore, the total D3-brane action is

$$\mathcal{S}_{D3} = \frac{N}{2\pi^2} \int dx_0 dx_1 dr d\psi \frac{1}{y^4} \left[\sqrt{\frac{r^2 + y^2\dot{\phi}^2}{1 + y'^2}} - r \right]. \quad (3.21)$$

Plugging in the D3-brane solution (3.8) one sees that the integrand vanishes $\mathcal{S}_{D3}|_{\text{on-shell}} = 0$.

It is illustrative to repeat the calculation of the D3-brane action in the second coordinate system (3.3), where now the field $u = u_0$ is a constant, so the variational principle for it is well defined. The classical on-shell action for the D3-brane is now given by

$$\mathcal{S}_{D3}|_{\text{on-shell}} = \frac{N}{2\pi^2} \int dx_0 dx_1 dz d\psi \frac{\cosh^2 u_0}{z^3} \left[\cosh u_0 \sqrt{\sinh^2 u_0 + \dot{\phi}^2} - \cosh^2 u_0 \right] = 0. \quad (3.22)$$

We conclude that the expectation value of a surface operator \mathcal{O}_Σ supported on $\Sigma = R^2$ in the probe approximation is given by

$$\langle \mathcal{O}_\Sigma \rangle = 1, \quad (3.23)$$

which agrees with the semiclassical gauge theory computation (2.15).

For the case of a surface operator \mathcal{O}_Σ supported on $\Sigma = S^2$, we find that the action of the dual D3-brane in the metric (3.12) is given by

$$\mathcal{S}_{D3} = \frac{N}{2\pi^2} \int d\rho d\vartheta d\varphi d\psi \cosh^3 u_0 \sinh^2 \rho \sin \vartheta \left[\sqrt{\sinh^2 u_0 + \dot{\phi}^2} - \cosh u_0 \right]. \quad (3.24)$$

In this case $u = u_0$ is also a constant and it satisfies the proper variational principle so after setting $\dot{\phi} = -1$ we find again that the on-shell action also vanishes $\mathcal{S}_{D3} = 0$. Thus the expectation value of a surface operator \mathcal{O}_Σ supported on $\Sigma = S^2$ in the probe approximation is given by

$$\langle \mathcal{O}_\Sigma \rangle = 1, \quad (3.25)$$

agreeing with the semiclassical gauge theory computation (2.15).

The conformal anomaly (2.9) for the surface operator \mathcal{O}_Σ with $\Sigma = R^2$ and S^2 vanishes in the probe approximation and due to (2.12) we have that $c_1 = 0$. It would be interesting to consider more general D3-brane solutions that end on a boundary with a different metric and on a general surface Σ . This would allow to compute the other anomaly coefficients c_i in (2.9).

3.3 Correlator with local operators in probe approximation

We now compute in the probe approximation the correlation function of a surface operator \mathcal{O}_Σ and the CPO's \mathcal{O}_Δ^I of dimension $\Delta = 2, 3$ in $\mathcal{N} = 4$ SYM. Analogous computations have been done for Wilson loops in [10, 31, 26] and for Wilson surfaces in six dimensions in [32, 33].

A CPO \mathcal{O}_Δ^I in $\mathcal{N} = 4$ SYM couples to the five dimensional scalar fluctuation s^I [34]. This fluctuation mode diagonalizes the linearized equations of motion of Type IIB supergravity expanded around $AdS_5 \times S^5$. It solves the Klein-Gordon equation in AdS_5 [35]

$$\nabla_\mu \nabla^\mu s^I = \Delta(\Delta - 4)s^I, \quad (3.26)$$

where Δ is the dimension of the dual CPO \mathcal{O}_Δ^I .

In the gauge theory (section 2.3) we saw that the only chiral primaries that have non-trivial coupling to the surface operators are the $SO(4)$ singlets. This can be seen here from the fact that in the metric (3.1) the brane is at $\theta = \pi/2$, where all the other spherical harmonics vanish. We can combine all the s^I which couple to the probe brane into one field on $AdS_5 \times S^5$ as

$$s = \sum_{\Delta} \sum_{k=-\Delta, -\Delta+2, \dots}^{\Delta} Y^{\Delta, k} \left(\theta = \frac{\pi}{2}, \phi \right) s^{\Delta, k}, \quad (3.27)$$

where $Y^{\Delta,k}$ are the SO(4) invariant spherical harmonics of SO(6) (see appendix A). When evaluated on the D3-brane they are given by

$$Y^{\Delta,k} \left(\theta = \frac{\pi}{2}, \phi \right) = C_{\Delta,k} e^{ik\phi}. \quad (3.28)$$

The correlator of a surface operator \mathcal{O}_Σ with \mathcal{O}_Δ^I is captured in the probe approximation by the solution of the linearized equations of motion of Type IIB supergravity for s^I in the presence of the D3-brane source. In order to calculate this we first need the bulk to boundary propagator in AdS_5 . Let us consider \mathcal{O}_Δ^I at a position (x'_0, x'_1, r', ψ') on the boundary of AdS_5 in the coordinate system (3.2). The propagator from the insertion point to a point in the bulk is given by

$$G = c \frac{y^\Delta}{((x - x')^2 + r^2 + r'^2 - 2rr' \cos(\psi - \psi') + y^2)^\Delta}, \quad (3.29)$$

where $c = \frac{\Delta+1}{2^{2-\Delta/2} N \sqrt{\Delta}}$ is chosen [10] such that the bulk computation of the two-point function of the dual CPO operator \mathcal{O}_Δ^I is unit normalized as in (2.27). We find it convenient to work in the coordinate system (3.3) where AdS_5 is foliated by $AdS_3 \times S^1$ slices. If we place \mathcal{O}_Δ^I at $(x'_0 = x'_1 = 0, \psi', z' = d)$ then the bulk to boundary propagator reads

$$G = c \frac{z^\Delta}{\cosh^\Delta u_0 (x_0^2 + x_1^2 + z^2 - 2dz \tanh u_0 \cos(\psi - \psi') + d^2)^\Delta}. \quad (3.30)$$

While the bulk field s^I has a simple propagator, it has rather complicated couplings to the usual supergravity fields. In the bulk it is related to the metric on AdS_5 , the metric on S^5 and the four forms on AdS_5 and S^5 . Following the notations of [35], the relevant fields sourced by s^I at linear order are

$$h_{\mu\nu}^{\text{AdS}}, \quad h_{\alpha\beta}^S, \quad a_{\mu\nu\rho\sigma}^{\text{AdS}}, \quad a_{\alpha\beta\gamma\delta}^S. \quad (3.31)$$

The fluctuation s^I sources these supergravity fields and these in turn couple to the probe D3-brane. To compute the one point function we need to expand the probe D3-brane action to linear order in the fluctuations (3.31). The linearized fluctuation contribution from the D3-brane DBI action is

$$\begin{aligned} \mathcal{L}_{\text{DBI}}^{(1)} &= \frac{T_{D3}}{2} \int \sqrt{\det(g)} g^{ab} (\partial_a X^\mu \partial_b X^\nu h_{\mu\nu}^{\text{AdS}} + \partial_a X^\alpha \partial_b X^\beta h_{\alpha\beta}^S) \\ &= \frac{T_{D3}}{2} \int dz dx_0 dx_1 d\psi \frac{\cosh^2 u}{z^3} \left(z^2 h_{zz}^{\text{AdS}} + z^2 h_{00}^{\text{AdS}} + z^2 h_{11}^{\text{AdS}} + h_{\psi\psi}^{\text{AdS}} + h_{\phi\phi}^S \right), \end{aligned} \quad (3.32)$$

where g is the induced metric.

The linearized fluctuation contribution from the WZ term of the D3-brane is:

$$\mathcal{L}_{\text{WZ}}^{(1)} = T_{D3} \int a_{\mu\nu\rho\sigma}^{\text{AdS}} \quad (3.33)$$

As can be seen from the expressions above, our probe D3-brane, which is mainly along AdS_5 , does not couple to linear order to $a_{\alpha\beta\gamma\delta}^S$, the 4-form on S^5 .

The last ingredient we need is the relation between the diagonal fluctuation s^I and the fluctuations in the basis (3.31). This is obtained by solving the linearized Type IIB supergravity equations of motion in the presence of the source $s = \sum_I s^I Y^I$, where Y^I is an $SO(6)$ spherical harmonic. The solution is given by [35, 36]

$$\begin{aligned} h_{\mu\nu}^{\text{AdS}} &= -\frac{6}{5}\Delta s g_{\mu\nu} + \frac{4}{\Delta+1}\nabla_{(\mu}\nabla_{\nu)}s, \\ h_{\alpha\beta}^S &= 2\Delta s g_{\alpha\beta}, \\ a_{\mu\nu\rho\sigma}^{\text{AdS}} &= -4\epsilon_{\mu\nu\rho\sigma\eta}\nabla^\eta s. \end{aligned} \quad (3.34)$$

where ∇_μ is the usual covariant derivative and $\nabla_{(\mu}\nabla_{\nu)}s$ denotes the symmetric traceless piece of $\nabla_\mu\nabla_\nu s$.

Finally we have all the pieces in place to calculate the correlator. We plug into (3.32) and (3.33) the expressions for the fluctuations (3.31) in terms of s using (3.34). For s we take the bulk to boundary propagator (3.30) with a boundary source

$$s_0 = \sum_{\Delta} \sum_{k=-\Delta, -\Delta+2, \dots}^{\Delta} Y^{\Delta,k} \left(\theta = \frac{\pi}{2}, \phi \right) s_0^{\Delta,k}, \quad (3.35)$$

The correlator of a surface operator \mathcal{O}_Σ with the $\mathcal{N} = 4$ SYM CPO $\mathcal{O}_{\Delta,k}$ in (2.28) is given by the derivative of the D3-brane action with respect to the source $s_0^{\Delta,k}(x)$. It is given by

$$\begin{aligned} \frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{\Delta,k} \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= -\frac{2^{1+\Delta/2}(\Delta+1)\sqrt{\Delta}T_{D3}d^2C_{\Delta,k}}{N \cosh^\Delta u_0} \int dz dx_0 dx_1 d\psi \\ &\quad \times \frac{z^{\Delta-1}e^{ik(\phi_0-\psi)}}{(x_0^2+x_1^2+z^2-2dz \tanh u_0 \cos(\psi-\psi')+d^2)^{\Delta+2}}, \end{aligned} \quad (3.36)$$

where we have used that on the D3-brane solution (3.9) $\psi + \phi = \phi_0$, where ϕ_0 is one of the parameters characterizing the D3-brane embedding. Integrating over x_0 and x_1 and using that $T_{D3} = N/2\pi^2$ (in the units $L^4 = 1$) yields

$$\frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{\Delta,k} \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \frac{2^{\Delta/2}\sqrt{\Delta}d^2C_{\Delta,k}}{\pi \cosh^\Delta u_0} \int_0^\infty dz \int_0^{2\pi} d\psi \frac{z^{\Delta-1}e^{ik(\phi_0-\psi)}}{(z^2 - 2dz \tanh u_0 \cos(\psi - \psi') + d^2)^{\Delta+1}}. \quad (3.37)$$

We make a change of variables from z to ζ defined by

$$\frac{2dz \tanh u_0}{z^2 + d^2} = \sqrt{1 - \frac{1}{\zeta^2}}. \quad (3.38)$$

As z varies between zero and infinity, ζ goes from one to $\cosh u_0$ (at $z = d$) and back to one. The resulting integral is then simplified by the fact that only terms with a branch-cut in ζ will contribute. This gives

$$\begin{aligned} \frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{\Delta,k} \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{\sqrt{\Delta}C_{\Delta,k}e^{ik(\phi_0-\psi')}}{2^{\Delta/2+1}\pi d^\Delta \sinh^\Delta u_0} 2 \int_1^{\cosh u_0} d\zeta \frac{\zeta \sinh u_0}{\sqrt{\cosh^2 u_0 - \zeta^2}} (\zeta^2 - 1)^{\Delta/2-1} \times \\ &\quad \times \int_0^{2\pi} d\tilde{\psi} \frac{e^{ik\tilde{\psi}}}{\left(\zeta - \sqrt{\zeta^2 - 1} \cos \tilde{\psi} \right)^{\Delta+1}}. \end{aligned} \quad (3.39)$$

We use now the integral representation of the associated Legendre polynomial¹¹

$$\int_0^{2\pi} d\tilde{\psi} \frac{e^{ik\tilde{\psi}}}{\left(\zeta - \sqrt{\zeta^2 - 1} \cos \tilde{\psi}\right)^{\Delta+1}} = 2\pi \frac{(\Delta - |k|)!}{\Delta!} P_{\Delta}^{|k|}(\zeta). \quad (3.40)$$

After a further change of variables

$$\sqrt{\zeta^2 - 1} = \sinh u_0 \sin \chi, \quad (3.41)$$

the correlator (3.39) reduces to the integral

$$\frac{(\Delta - k)! \sqrt{\Delta} C_{\Delta,k} e^{ik(\phi_0 - \psi')}}{\Delta! 2^{\Delta/2} d^{\Delta}} \int_0^{\pi} d\chi \sin^{\Delta-1} \chi P_{\Delta}^{|k|} \left(\sqrt{1 + \sinh^2 u_0 \sin^2 \chi} \right). \quad (3.42)$$

We can now extract the correlators of the surface operator \mathcal{O}_{Σ} with any CPO $\mathcal{O}_{\Delta,k}$. We write down explicitly the result for $\Delta = 2$ and 3 which we also did in the gauge theory in section 2 and do in the “bubbling” geometry description in the next section. Using the spherical harmonics in appendix A we have that $C_{2,0} = 1/\sqrt{3}$, $C_{2,\pm 2} = 1/2$, $C_{3,\pm 1} = \sqrt{3}/4$ and $C_{3,\pm 3} = 1/2\sqrt{2}$ while the relevant associated Legendre polynomials are given by $P_2^0(x) = (3x^2 - 1)/2$, $P_2^2(x) = 3(x^2 - 1)$, $P_3^1(x) = 3\sqrt{x^2 - 1}(5x^2 - 1)/2$ and $P_3^3(x) = 15(x^2 - 1)^{3/2}$. Performing the integral in (3.42) we conclude that the correlator of \mathcal{O}_{Σ} with the CPO’s (2.29) with $\Delta = 2$ and 3 are given in the probe approximation by

$$\begin{aligned} \frac{\langle \mathcal{O}_{2,0} \cdot \mathcal{O}_{\Sigma} \rangle}{\langle \mathcal{O}_{\Sigma} \rangle} &= \frac{2}{\sqrt{6}} \frac{\cosh^2 u_0}{d^2}; & \frac{\langle \mathcal{O}_{2,2} \cdot \mathcal{O}_{\Sigma} \rangle}{\langle \mathcal{O}_{\Sigma} \rangle} &= \frac{1}{\sqrt{2}} \frac{\sinh^2 u_0}{d^2} e^{2i(\phi_0 - \psi')} \\ \frac{\langle \mathcal{O}_{3,1} \cdot \mathcal{O}_{\Sigma} \rangle}{\langle \mathcal{O}_{\Sigma} \rangle} &= \frac{1}{\sqrt{2}} \frac{\cosh^2 u_0 \sinh u_0}{d^3} e^{i(\phi_0 - \psi')}; & \frac{\langle \mathcal{O}_{3,3} \cdot \mathcal{O}_{\Sigma} \rangle}{\langle \mathcal{O}_{\Sigma} \rangle} &= \frac{1}{\sqrt{3}} \frac{\sinh^3 u_0}{d^3} e^{3i(\phi_0 - \psi')}. \end{aligned} \quad (3.43)$$

The remaining correlators can be obtained by complex conjugating (3.43).

Recall that the probe calculation is done with a single D3-brane and is dual to the surface operator \mathcal{O}_{Σ} in the gauge theory where the gauge group is broken to $U(1) \times SU(N - 1)$. In order to compare to the gauge theory result (2.30) we take $N_1 = 1$ and $N_2 = N - 1$, we use the identification (3.10) to relate $e^{i\phi_0} \sinh u_0$ to $\beta_1 + i\gamma_1$, and take $\beta_2 = \gamma_2 = 0$. Then with $z = d e^{i\psi'}$, we find that (3.43) agrees precisely with (2.30) for $\mathcal{O}_{2,2}$ and $\mathcal{O}_{3,\pm 3}$. The other correlators in (3.43) exactly reproduce the semiclassical result in (2.30) but deviate from the classical result of the gauge theory by a quantum correction, proportional to the ’t Hooft coupling λ . The general form of the correlator (3.42) for $\mathcal{O}_{\Delta,k}$ computed in the probe approximation indicates that the quantum corrections to the semiclassical gauge theory computation truncate to order $\lambda^{(\Delta - |k|)/2}$, suggesting that the gauge theory description of these correlators may be captured exactly by a reduced matrix model, which has a finite number of loop corrections.¹²

¹¹We use a slightly modified definition, $P_{\Delta}^{|k|}(x) = \frac{1}{2^{\Delta} \Delta!} (x^2 - 1)^{|k|/2} (d^{\Delta+|k|} (x^2 - 1)^{\Delta} / d^{\Delta+|k|} x)$ which is single valued for argument greater than one.

¹²This conjecture further requires that there are no $1/\sqrt{\lambda}$ correction for these observables in supergravity.

3.4 Correlator with Wilson loop operator

We wish to examine the correlator of the surface operator \mathcal{O}_Σ with the Wilson loop operator (2.38). We employ the target-space metric (3.12) where the brane is at fixed $u = u_0$. The Wilson loop is described, as usual, by a semiclassical string surface [37, 38], ending on the boundary of $AdS_5 \times S^5$ along a path related to the specific choice of loop. The specific loops we are considering are those studied in [15, 23], where the relevant boundary conditions are further explained. The string worldsheet is not at fixed u , and it varies from ∞ to $u = u_0$, where it ends on the D3-brane probe. In the $u \rightarrow \infty$ region the string worldsheet ends in the $AdS_5 \times S^5$ boundary along the desired loop, which is a circle along the ψ direction at fixed¹³ ρ . At the same time the string also wraps the direction ϕ along a circle of S^5 at $\theta = \theta_0$ in the metric (3.1). The extra parameter ψ_0 which appeared in the definition of the loop is the relative phase between the two angles, so that at the boundary $\psi + \phi = \psi_0$, matching the phase of the Wilson loop operator in (2.38).

Finding the string solution for this configuration is by now a standard procedure. With the worldsheet coordinates σ and τ one assumes the rotationally-symmetric ansatz

$$u = u(\sigma), \quad \psi = \tau + \alpha(\sigma), \quad \theta = \theta(\sigma), \quad \phi = -\tau - \zeta(\sigma), \quad (3.44)$$

and one can consistently keep all the other fields constant. The boundary conditions at $u \rightarrow \infty$ on ψ and ϕ are such that $\alpha(0) - \zeta(0) = \psi_0$, so that at the boundary we get $\psi + \phi = \psi_0$, which is the phase corresponding to the Wilson loop (2.38). These extra two degrees of freedom are required in order to allow the relative phase of ψ and ϕ to change as the string moves in the u direction. At the other end of the string worldsheet, where the string ends on the probe D3-brane, we have that $\psi + \phi = \phi_0$ (3.8), so $\alpha - \zeta = \phi_0$ there.

The string Lagrangian in the conformal gauge is

$$\mathcal{L} = \frac{\sqrt{\lambda}}{4\pi} (u'^2 + \sinh^2 u (1 + \alpha'^2) + \theta'^2 + \sin^2 \theta (1 + \zeta'^2)). \quad (3.45)$$

Two conserved quantities are the momenta conjugate to α and ζ

$$p_\alpha = \sinh^2 u \alpha', \quad p_\zeta = \sin^2 \theta \zeta'. \quad (3.46)$$

There are two Virasoro constraints

$$\sinh^2 u \alpha' + \sin^2 \theta \zeta' = 0, \quad u'^2 - \sinh^2 u (1 - \alpha'^2) + \theta'^2 - \sin^2 \theta (1 - \zeta'^2) = 0. \quad (3.47)$$

The first one sets $p_\alpha = -p_\zeta$. By solving the equations of motion for u and θ one finds expressions similar to the second Virasoro constraint for each independently

$$u'^2 - \sinh^2 u + \frac{p_\alpha^2}{\sinh^2 u} = a^2, \quad \theta'^2 - \sin^2 \theta + \frac{p_\zeta^2}{\sin^2 \theta} = -a^2. \quad (3.48)$$

These equations can be solved in general in terms of elliptic integrals, which we will not do here, since we will use in section 4.4 a different coordinate system where the solution can be found rather easily.

¹³Due to conformal invariance the radius of the loop in the gauge theory, and the value of ρ here are inconsequential

Instead, let us focus for now on the simple case with $p_\alpha = -p_\zeta = 0$ which is the case when the Wilson loop (2.38) and surface operator \mathcal{O}_Σ are in phase, i.e. $\psi_0 = \phi_0$. One solution is for $a^2 = 1$ which corresponds to the Wilson loop on a great circle on S^5 .

$$\psi - \psi_0 = -\phi = \tau, \quad \cosh u = \frac{1}{\cos \sigma}, \quad \theta = \frac{\pi}{2}. \quad (3.49)$$

The coordinate u varies between infinity, the boundary of spacetime, and u_0 where it ends on the D3-brane. After subtracting the divergence from infinity, the finite part of the string action is

$$\mathcal{S} = \frac{\sqrt{\lambda}}{2} \int d\sigma (u'^2 + \cosh^2 u) = -\sqrt{\lambda} \sinh u_0. \quad (3.50)$$

On the D3-brane worldvolume we have a non-trivial holonomy for the worldvolume gauge field $A_\psi = \alpha$ (3.10). Since the string ends on the D3-brane along a curve wrapping the ψ direction there is a contribution from a boundary term on the worldsheet. It is given by

$$e^{i \oint A_\psi d\psi} = e^{2\pi i \alpha}. \quad (3.51)$$

The correlator between a surface operator \mathcal{O}_Σ and the Wilson loop is given in the semiclassical probe approximation by string worldsheet action. Combining the bulk and boundary term and using the relation (3.10) between $\sinh u_0$ and the parameters of the surface operator \mathcal{O}_Σ in the gauge theory, the expectation value of the Wilson loop in the presence of the surface operator \mathcal{O}_Σ is given by

$$\frac{\langle W_{\pi/2, \phi_0} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \exp(2\pi|\beta + i\gamma| + 2\pi i \alpha). \quad (3.52)$$

This probe D3-brane computation reproduces the result of the gauge theory computation (2.41) for gauge group $U(1) \times SU(N-1)$.

Note that there is another string solution with these boundary conditions [22], which has $a = 0$ and

$$\sinh u = \frac{1}{\sinh \sigma}, \quad \sin \theta = \frac{1}{\cosh \sigma}. \quad (3.53)$$

This solution does not end on the D3-brane, but is still a solution with the prescribed boundary conditions. This solution has zero action.

Using the coordinates in the next section and the explicit solution in section 4.4, we can evaluate the action for arbitrary $\psi_0 - \phi_0$ and θ_0 . There are two disconnected solutions, found already in [15] with action

$$\mathcal{S} = \pm \sqrt{\lambda} \cos \theta_0. \quad (3.54)$$

Clearly the solution with a negative sign dominates, but the other one, which is unstable, can also be found in the asymptotic expansion of the Gaussian matrix model and we therefore retain it.

The connected solution, with the contribution of the holonomy on the probe D3-brane has action

$$\mathcal{S} = -\sqrt{\lambda} \sin \theta_0 \cos(\psi_0 - \phi_0) \sinh u_0 - 2\pi i \alpha. \quad (3.55)$$

Summing over the three saddle points and using (3.10) we get the result

$$\frac{\langle W_{\theta_0, \psi_0} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \exp \left(2\pi \sin \theta_0 (\beta \cos \psi_0 + \gamma \sin \psi_0) + 2\pi i \alpha \right) + \exp \left(\sqrt{\lambda} \cos \theta_0 \right) + \exp \left(-\sqrt{\lambda} \cos \theta_0 \right). \quad (3.56)$$

This agrees precisely with the gauge theory prediction (2.41) for $N_1 = 1$, $N_2 = N - 1$ and $\beta_2 = \gamma_2 = 0$ in the large N limit, including the last term, corresponding to an unstable solution.

In string theory the difference between a Wilson loop and an 't Hooft loop is simply the replacement of a fundamental string by a D-string. Geometrically the surfaces will look the same with the only difference being a relative factor of $1/g_s = 4\pi/g_{\text{YM}}^2$ in the on-shell action (3.50). In this case the D-strings couples through its boundary to the dual gauge field on the probe D3-brane, which is now proportional to η (3.10). Therefore the final answer of the calculation of the correlator between a surface operator \mathcal{O}_Σ and an 't Hooft operator in the probe approximation is

$$\frac{\langle T_{\theta_0, \psi_0} \cdot \mathcal{O}_\Sigma \rangle}{\langle \mathcal{O}_\Sigma \rangle} = \exp \left(\frac{8\pi^2}{g_{\text{YM}}^2} \sin \theta_0 (\beta \cos \psi_0 + \gamma \sin \psi_0) + 2\pi i \eta \right) + \exp \left(\frac{4\pi N}{\sqrt{\lambda}} \cos \theta_0 \right) + \exp \left(-\frac{4\pi N}{\sqrt{\lambda}} \cos \theta_0 \right). \quad (3.57)$$

This calculation done with a D1-brane is exactly the S -dual of (3.56) accounting also for the S duality transformation $\alpha \rightarrow \eta$.

4. “Bubbling” surface operators in $\mathcal{N} = 4$ SYM

In this section we perform calculations with surface operators in $\mathcal{N} = 4$ SYM using the Type IIB supergravity solutions proposed in [7] as the gravitational description of surface operators. We begin with a summary of the main features of the supergravity solutions. We then review how to extract correlation functions of gauge theory operators from an asymptotically $AdS_5 \times S^5$ bulk solution and summarize the main formulas. We then calculate using the supergravity solutions the correlators of a surface operator \mathcal{O}_Σ with local operators, with a Wilson and an 't Hooft loop.

4.1 Surface operator as “bubbling” supergravity solution

In [7] the gravitational description of all the maximally supersymmetric operators \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM was found in terms of smooth ten dimensional solutions of Type IIB supergravity which are asymptotically $AdS_5 \times S^5$. These supergravity solutions capture the complete backreaction of the configuration of D3 branes in $AdS_5 \times S^5$ considered in the previous section, which describe a surface operator \mathcal{O}_Σ in the probe approximation. Here we summarize the main ingredients of the solutions needed for the holographic computation of the correlation functions. See [7] for more details about the solution and the identification with the maximally supersymmetric surface operators in $\mathcal{N} = 4$ SYM.

By analyzing the symmetries of the maximally supersymmetric surface operators in $\mathcal{N} = 4$ SYM one may observe [7] that the corresponding ansatz for the dual supergravity solution can be found by performing a double analytic continuation of the LLM ansatz [8, 9] dual to the maximally supersymmetric local operators in $\mathcal{N} = 4$ SYM. The Einstein frame metric describing a surface operator in $\mathcal{N} = 4$ SYM is given by

$$ds^2 = y \sqrt{\frac{2z+1}{2z-1}} ds_{AdS_3}^2 + y \sqrt{\frac{2z-1}{2z+1}} d\Omega_3 + \frac{2y}{\sqrt{4z^2-1}} (d\chi + V)^2 + \frac{\sqrt{4z^2-1}}{2y} (dy^2 + dx_i dx_i), \quad (4.1)$$

while the RR five-form field strength is given by

$$F_5 = \frac{1}{4} \left(d \left[y^2 \frac{2z+1}{2z-1} (d\chi + V) \right] - y^3 *_3 d \left[\frac{z + \frac{1}{2}}{y^2} \right] \right) \wedge d \text{Vol}_{AdS_3} - \frac{1}{4} \left(d \left[y^2 \frac{2z-1}{2z+1} (d\chi + V) \right] - y^3 *_3 d \left[\frac{z - \frac{1}{2}}{y^2} \right] \right) \wedge d\Omega_3. \quad (4.2)$$

z is a function on the space X parametrized by x_1, x_2 and y , with $ds_X^2 = dy^2 + dx_i dx_i$ and $y \geq 0$. V is a one-form satisfying $dV = \frac{1}{y} *_X dz$, and hence, the metric and the five-form are completely determined by $z(x_1, x_2, y)$.

This supergravity ansatz describes a surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ on $AdS_3 \times S^1$, as the metric in the conformal boundary of (4.1) is that of $AdS_3 \times S^1$. The choice of AdS_3 metric determines the surface Σ on which the surface operator \mathcal{O}_Σ is supported: AdS_3 in Poincaré coordinate corresponds to a surface operator \mathcal{O}_Σ supported on $\Sigma = R^2$ while global AdS_3 corresponds to a surface operator \mathcal{O}_Σ supported on $\Sigma = S^2$.

A non-trivial solution to the equations of motion is obtained by specifying a configuration of M point-like particles in X . The data from which the solution is determined is the particle positions (\vec{x}_l, y_l) in X (see figure 1). Given a particle distribution, the function $z(x_1, x_2, y)$ is found by solving the following differential equation¹⁴

$$\partial_i \partial_i z(x_1, x_2, y) + y \partial_y \left(\frac{\partial_y z(x_1, x_2, y)}{y} \right) = \sum_{l=1}^M 2\pi y_l \delta(y - y_l) \delta^{(2)}(\vec{x} - \vec{x}_l). \quad (4.3)$$

This equation is equivalent to the six dimensional Laplace equation with $SO(4)$ invariant sources for the variable

$$\Phi = \frac{z}{y^2}. \quad (4.4)$$

The solution to (4.3) giving rise to a non-singular smooth metric is given by

$$z(x_1, x_2, y) = \frac{1}{2} + \sum_{l=1}^M z_l(x_1, x_2, y), \quad (4.5)$$

where

$$z_l(x_1, x_2, y) = \frac{(\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2}{2\sqrt{((\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2)^2 - 4y_l^2 y^2}} - \frac{1}{2}. \quad (4.6)$$

¹⁴The “charge” $Q_l = 2\pi y_l$ associated to these particles is fixed such that the χ circle shrinks smoothly at $y = y_l$.

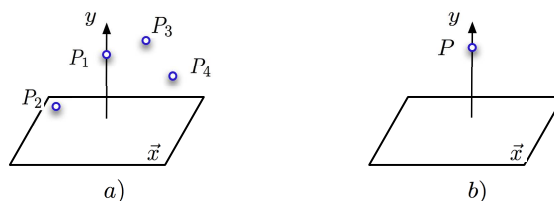


Figure 1: a) The metric and five-form flux is determined once the position of the particles in X – labeled by coordinates (\vec{x}_l, y_l) where $y \geq 0$ – is given. The l -th particle is associated with a point $P_l \in X$. b) The configuration corresponding to the $AdS_5 \times S^5$ vacuum.

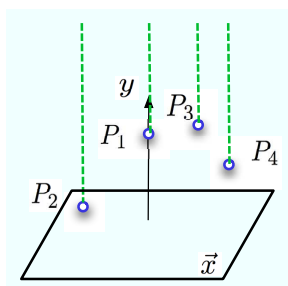


Figure 2: A disk D can be constructed by fibering S^1 over an interval connecting the “charge” at the point $P_l \in X$ with (\vec{x}_l, y_l) coordinates and the boundary of $AdS_5 \times S^5$.

The one form $V \equiv V_I dx^I$ can be determined up to an exact form by solving $dV = \frac{1}{y} *_X dz$

$$V_I = -\epsilon_{IJ} \sum_{l=1}^M \frac{(x^J - x_l^J)((\vec{x} - \vec{x}_l)^2 + y^2 - y_l^2)}{2(\vec{x} - \vec{x}_l)^2 \sqrt{((\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2)^2 - 4y_l^2 y^2}}. \quad (4.7)$$

Therefore, the metric and five-form field strength are determined by the integer M and by the particle locations (\vec{x}_l, y_l) for $l = 1, \dots, M$.

As explained in [7], further data needs to be given in order to fully specify a Type IIB supergravity solution. These metrics have M non-trivial two-cycles D_l and a solution is fully specified only after we fix the periods of the NS-NS and RR two-form gauge fields on the two cycles

$$\int_{D_l} \frac{B_{\text{NS}}}{2\pi}; \quad \int_{D_l} \frac{B_R}{2\pi}; \quad l = 1, \dots, M. \quad (4.8)$$

Therefore, these “bubbling” supergravity solutions depend on $4M$ real parameters and there is a one to one correspondence between these supergravity solutions and the maximally supersymmetric surface operators in $\mathcal{N} = 4$ SYM, which also depend on $4M$

parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$. The explicit mapping is given by [7]

$$\begin{aligned}\alpha_l &= - \int_{D_l} \frac{B_{\text{NS}}}{2\pi} \\ \beta_l + i\gamma_l &= \frac{x_{l,1} + ix_{l,2}}{2\pi l_s^2} \\ \eta_l &= \int_{D_l} \frac{B_R}{2\pi} \\ y_l^2 &= \frac{N_l}{N}.\end{aligned}\quad l = 1, \dots, M \tag{4.9}$$

The ranks of the subgroups of the gauge group $L = \prod_{l=1}^M \text{U}(N_l)$ left unbroken by \mathcal{O}_Σ is encoded in the positions in the y axis of the M particles via $N_l = \frac{y_l^2}{4\pi l_p^4}$, as y_l^2 determines the amount of five-form flux through the corresponding S^5 . In the units where the $AdS_5 \times S^5$ radius of curvature $L = 1$, we have that $y_l^2 = N_l/N$.

These solutions are regular but become singular precisely when the path integral describing a surface operator \mathcal{O}_Σ becomes singular [7]. This occurs when the unbroken gauge group near Σ is enhanced, which corresponds in the supergravity solution to colliding the charges in figure 1. When two charges collide an S^2 in the geometry shrinks to zero size and the solution is singular.

4.2 Gauge theory correlation functions from ten dimensional “bubbling” solutions

Here we summarize the main ideas and formulas involved in extracting the correlation function of local operators in the boundary gauge theory from the ten dimensional “bubbling” supergravity solution. We refer to [39] for more details and derivations.

The asymptotically $AdS_5 \times S^5$ solutions of Type IIB supergravity describing maximally supersymmetric surface operators \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM are ten dimensional. Holography is on the other hand an equivalence between quantum gravity with AdS_5 boundary conditions and a four dimensional field theory on the boundary [40, 41, 34]. Therefore, given a ten dimensional asymptotically $AdS_5 \times S^5$ solution of Type IIB supergravity, one must systematically reduce the ten dimensional solution to a solution of five dimensional gravity coupled to — in general — an infinite number of five dimensional fields. Once this reduction and a local five dimensional action has been found, one can apply the usual holographic rules [40, 41, 34] and use the bulk five dimensional description to compute the correlation function of boundary operators. In order to get the renormalized correlation functions in the gauge theory one must appropriately renormalize the bulk gravitational action using holographic renormalization [42–44] (for a review and more references see [45]).

The first step in constructing the five dimensional gravity action from which one can holographically compute gauge theory correlation functions is to decompose the ten dimensional solution into $AdS_5 \times S^5$ plus the deviation of the solution away from $AdS_5 \times S^5$. One then expands the deviation in a complete basis of $\text{SO}(6)$ spherical harmonics which yields an infinite collection of five dimensional fluctuation fields. The decomposition into

harmonics of the the metric and five-form deviations is given by

$$\begin{aligned}
 h_{\mu\nu} &= \sum h_{\mu\nu}^{I_1}(x) Y^{I_1}(y) \\
 h_{\mu a}(x, y) &= \sum (B_{(v)\mu}^{I_5}(x) Y_a^{I_5}(y) + B_{(s)\mu}^{I_1}(x) D_a Y^{I_1}(y)) \\
 h_{(ab)}(x, y) &= \sum (\hat{\phi}_{(t)}^{I_{14}}(x) Y_{(ab)}^{I_{14}}(y) + \phi_{(v)}^{I_5}(x) D_{(a} Y_{b)}^{I_5}(y) + \phi_{(s)}^{I_1}(x) D_{(a} D_{b)} Y^{I_1}(y)) \\
 h_a^a(x, y) &= \sum \pi^{I_1}(x) Y^{I_1}(y),
 \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
 f_{abcd\mu}(x, y) &= \sum (D_\mu b_{(s)}^{I_1}(x) \epsilon_{abcd}{}^e D_e Y^{I_1}(y) + (\Lambda^{I_5} - 4) b_\mu^{I_5}(x) \epsilon_{abcd}{}^e Y_e^{I_5}(y)) \\
 f_{abcde}(x, y) &= \sum b_{(s)}^{I_1}(x) \Lambda^{I_1} \epsilon_{abcde} Y^{I_1}(y),
 \end{aligned} \tag{4.11}$$

where x^μ are AdS_5 coordinates while y^a are S^5 coordinates. Y^{I_1} , $Y_a^{I_5}$ and $Y_{(ab)}^{I_{14}}$ are scalar, vector and symmetric traceless $SO(6)$ spherical harmonics respectively while $-\Lambda^{I_1}$ and $-\Lambda^{I_5}$ are the eigenvalues of the d'Alembertian on scalar and vector harmonics (see appendix A for more details on the relevant spherical harmonics). One must then solve the ten dimensional Type IIB supergravity equations of motion perturbatively in the number of fluctuations.

In [35] the linearized analysis was performed in the de Donder gauge which retains the physical fluctuations. For our solutions it is convenient to work with gauge invariant variables as our “bubbling” solution (4.1) is not in the de Donder gauge. The perturbative analysis using gauge invariant variables has been carried out by Skenderis and Taylor in [46], where they showed that the equations of motion for the gauge invariant variables indeed coincide with the equations of motion in the de Donder gauge if the fluctuations in the de Donder gauge are replaced by their gauge invariant counterparts. The gauge invariant variables to linear order in the fluctuations are given in terms of the fluctuations in (4.10), (4.11) by [39]

$$\begin{aligned}
 \hat{\pi}^{I_1} &= \pi^{I_1} - \Lambda^{I_1} \phi_{(s)}^{I_1} \\
 \hat{b}^{I_1} &= b_{(s)}^{I_1} - \frac{1}{2} \phi_{(s)}^{I_1} \\
 \hat{B}_{(v)\mu}^{I_5} &= B_{(v)\mu}^{I_5} - \frac{1}{2} D_\mu \phi_{(v)}^{I_5} \\
 \hat{b}_\mu^{I_5} &= b_\mu^{I_5} - \frac{1}{2(\Lambda^{I_5} - 4)} D_\mu \phi_{(v)}^{I_5} \\
 \hat{\phi}^{I_{14}} &= \phi^{I_{14}}.
 \end{aligned} \tag{4.12}$$

The gauge invariant fluctuations that diagonalize the equations of motion to leading order in fluctuations and that couple to the CPO's \mathcal{O}_Δ^I with $\Delta = 2, 3$, the $SO(6)$ R -symmetry current J_m^a and the stress-energy tensor T_{mn} of $\mathcal{N} = 4$ SYM are [34]

$$\begin{aligned}
 \hat{s}^{2I} &= \frac{1}{80} (\hat{\pi}^{2I} - 60 \hat{b}^{2I}) \\
 \hat{s}^{3I} &= \frac{1}{100} (\hat{\pi}^{3I} - 70 \hat{b}^{3I})
 \end{aligned}$$

$$\begin{aligned} A_\mu^a &= \hat{B}_{(v)\mu}^0 - 12\hat{b}_\mu^0 \\ \tilde{h}_{\mu\nu} &= h_{\mu\nu}^0 + \frac{1}{3}g_{\mu\nu}^0\pi^0. \end{aligned} \quad (4.13)$$

In order to calculate correlation functions in the boundary gauge theory from the bulk description one must find a local five dimensional action that controls the dynamics of the gauge invariant fluctuations dual to the gauge theory operators. This requires finding a non-linear map between local five dimension fields and the gauge invariant fluctuations obtained by expanding the ten dimensional equations of motion. This mapping is constructed such that a solution of the five dimensional equations of motion yields a solution of the ten dimensional Type IIB supergravity equations of motion. This mapping can be constructed perturbatively in the number of fluctuations. This non-linear map between local fields and gauge invariant fluctuations has been constructed in [46].

Once a local five dimensional action yielding the appropriate five dimensional equations of motion is constructed, the correlation functions of renormalized boundary operators can be computed using holographic renormalization. In order to construct the renormalized correlation function of boundary operators the bulk action must be enriched with a set of boundary terms to make the on-shell bulk action finite. Despite the fact that the five dimensional action in general contains an infinite number of fields, only a finite number of terms in the action are infrared divergent. It is only the bulk fields dual to low dimension operators in the gauge theory that require boundary counterterms. Once a complete set of counterterms for the five dimensional action are constructed, one can calculate the correlation functions of boundary operators by taking functional derivatives of the renormalized bulk gravitational action.

Here we are interested in computing the correlation function of the CPO's \mathcal{O}_Δ^I with $\Delta = 2, 3$, of the R -symmetry current J_m^a and of the stress-energy tensor T_{mn} with a surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM. The bulk action that captures the degrees of freedom dual to these operators is given by [43, 46]¹⁵

$$\begin{aligned} S_{\text{bulk}} = \frac{N^2}{2\pi^2} \int d^5x \sqrt{-G} \left(\frac{R}{4} + \frac{1}{2}G^{\mu\nu}\partial_\mu\Theta\partial_\nu\Theta + \frac{1}{2}G^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - 2\Theta^2 - \frac{3}{2}\Psi^2 \right. \\ \left. - \frac{4\sqrt{6}}{3}\Theta^3 - \frac{1}{16}F_{\mu\nu}^a F_{\mu\nu}^a \right), \end{aligned} \quad (4.14)$$

where $F_{\mu\nu}^a$ is the field-strength of a vector potential A_μ^a and $G_{\mu\nu}$, Θ , Ψ and A_μ^a couple to T_{mn} , $\mathcal{O}_{\Delta=2}^I$, $\mathcal{O}_{\Delta=3}^I$ and J_m^a respectively and a is an $\text{SO}(6)$ vector index. The boundary counterterms required to have a finite on-shell action and from which the correlation function of renormalized operators can be computed can be found by solving the equations of motion of (4.14) near the boundary of AdS_5 . They are written down in [43, 46].

In order to calculate the one point function of these local operators using the bulk description it is convenient to write the five dimensional metric $G_{\mu\nu}$ in the Fefferman-

¹⁵This action is written, as throughout the rest of the paper, in units where the radius of curvature of $AdS_5 \times S^5$ is $L^4 = 4\pi g_s N l_s^4 = 1$.

Graham form

$$ds_5^2 = \frac{dU^2}{U^2} + \frac{1}{U^2} (G_{(0)mn} + U^2 G_{(2)mn} + \dots) dx^m dx^n, \quad (4.15)$$

where in our case $G_{(0)}$ is the $AdS_3 \times S^1$ metric on the conformal boundary at $U = 0$, where $\mathcal{N} = 4$ SYM lives. The bulk fields Θ, Ψ and A_μ also have a near boundary expansion. It is given by¹⁶

$$\begin{aligned} \Theta &\equiv \hat{s}^{2I}(x, U) = U^2 [\hat{s}^{2I}]_2 Y_2^I + \dots \\ \Psi &\equiv \hat{s}^{3I}(x, U) = U^3 [\hat{s}^{3I}]_3 Y_3^I + \dots \\ A_\mu^a(x, U) &= U^2 [A_\mu^a]_2(x) Y^{1a} + \dots, \end{aligned} \quad (4.16)$$

where $[\cdot]$ is the normalizable mode of the field.

Evaluation of the on-shell one point function of the renormalized bulk action shows that the vacuum expectation value of these operators is captured by the normalizable mode of the dual bulk field. In order to calculate the one point function of a unit normalized operator, we must divide the one point function obtained from supergravity by the square root of the two-point function obtained from supergravity [36], such that the supergravity computation of the vacuum expectation value corresponds to that of a unit normalized operator with two-point function given by (2.27). After taking this into account, we have that the one point function of the CPO's \mathcal{O}_Δ^I with $\Delta = 2, 3$ and of the R -symmetry current are given by [43, 46]¹⁷

$$\begin{aligned} \langle \mathcal{O}_2^I \rangle_{\mathcal{O}_\Sigma} &= \frac{N}{2} \frac{2\sqrt{8}}{3} [\hat{s}^{2I}]_2, \\ \langle \mathcal{O}_3^I \rangle_{\mathcal{O}_\Sigma} &= \frac{3N}{2\sqrt{6}} [\hat{s}^{3I}]_3, \\ \langle J_\mu^a \rangle_{\mathcal{O}_\Sigma} &= -\frac{\sqrt{2}N^2}{24\pi^2} [A_\mu^a]_2. \end{aligned} \quad (4.17)$$

The computation of the vacuum expectation value of the stress-energy tensor T_{mn} is more subtle as the bulk scalar field $\Theta(x, U)$ enters in the renormalization procedure for the five dimensional metric. The formula for the vacuum expectation value of the stress-energy tensor is given by [43, 46]

$$\begin{aligned} \langle T_{mn} \rangle_{\mathcal{O}_\Sigma} &= \frac{N^2}{2\pi^2} \left(g_{(4)mn} - \frac{2}{9} [\hat{s}^{2I}]_2 [\hat{s}^{2I}]_2 g_{(0)mn} \right. \\ &\quad \left. + \frac{1}{8} [\text{Tr} g_{(2)}^2 - (\text{Tr} g_{(2)})^2] g_{(0)mn} - \frac{1}{2} (g_{(2)}^2)_{mn} + \frac{1}{4} g_{(2)mn} \text{Tr} g_{(2)} + \frac{3}{2} h_{(4)mn} \right). \end{aligned} \quad (4.18)$$

In this formula $g_{(\cdot)mn}$ denotes the various terms in the near boundary expansion of the ten dimensional metric in the Graham-Fefferman coordinate U in (4.15).

¹⁶In our solutions the bulk fields do not have a source term and therefore we do not write it. In a general solution a non-normalizable mode has to be added to these asymptotic solutions.

¹⁷The current J_μ^a is normalized so that it has the canonical two-point function [43].

As explained in section 2, the expectation value of a local operator $\langle \mathcal{O} \rangle_{\mathcal{O}_\Sigma}$ in a non-trivial state of the gauge theory on $AdS_3 \times S^1$ corresponds to the two-point correlation function of the surface operator \mathcal{O}_Σ with the local operator \mathcal{O} in $\mathcal{N} = 4$ SYM on R^4

$$\frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O} \rangle}{\langle \mathcal{O}_\Sigma \rangle}. \quad (4.19)$$

Therefore, we can calculate these correlators computed in the previous sections in other regimes by using the supergravity description of surface operators.

4.3 Correlator of surface operators with local operators from “bubbling” solutions

We can now compute the correlation function of a surface operator \mathcal{O}_Σ with the CPO’s \mathcal{O}_Δ^I of $\Delta = 2, 3$, with the $U(1)$ current J_m^ψ and with the stress-energy tensor T_{mn} of $\mathcal{N} = 4$ SYM using the holographic correspondence. The analogous calculation for local operators has been carried out by Skenderis and Taylor in [39]. As summarized in the previous subsection, the computation of the correlator of \mathcal{O}_Σ with these local operators requires extracting the appropriate coefficient of the near boundary expansion of the corresponding fluctuation mode in (4.17), (4.18) from the “bubbling” supergravity solution (4.1), (4.2).

Before looking at the fluctuations, we write down a reference $AdS_5 \times S^5$ solution, which is the metric (4.1), (4.2) with a single particle source. If the particle position is at $(\vec{x}^{(0)}, y^{(0)} = 1)$, then in the coordinates

$$\begin{aligned} x^1 - x_1^{(0)} + i(x^2 - x_2^{(0)}) &= r e^{i\alpha} \\ r &= R \sin \theta \\ y &= \sqrt{1 + R^2} \cos \theta \\ \chi &= \frac{1}{2}(\psi - \phi) \\ \alpha &= \psi + \phi, \end{aligned} \quad (4.20)$$

we have that:

$$\begin{aligned} z^{(0)} &= \frac{1}{2} \frac{R^2 + \cos^2 \theta + 1}{R^2 + \sin^2 \theta} \\ V^{(0)} &= \frac{1}{2} \frac{R^2 - \sin^2 \theta}{R^2 + \sin^2 \theta} d\alpha. \end{aligned} \quad (4.21)$$

Inserting (4.21) into (4.1) results in the $AdS_5 \times S^5$ metric with AdS_5 foliated by $AdS_3 \times S^1$ slices¹⁸

$$ds^2 = \left[(1 + R^2) ds_{AdS_3}^2 + \frac{dR^2}{1 + R^2} + R^2 d\psi^2 + \cos^2 \theta d\Omega_3 + d\theta^2 + \sin^2 \theta d\phi^2 \right]. \quad (4.22)$$

The five-form flux (4.2) is given by

$$F_5 = R(R^2 + 1) dR \wedge d\psi \wedge dVol_{AdS_3} + \cos^3 \theta \sin \theta d\phi \wedge d\theta \wedge d\Omega_3. \quad (4.23)$$

¹⁸This is the same as (3.12) with $R = \sinh u$.

The supergravity solution (4.1), (4.2) describing a maximally supersymmetric surface operator \mathcal{O}_Σ is characterized by a choice of particle distribution (\vec{x}_l, y_l) with $l = 1, \dots, M$. This determines the function $z(x_1, x_2, y)$ and the one form V . For computational purposes we find it useful to use the function Φ (4.4), which is given in terms of $z(x_1, x_2, y)$ by

$$\Phi = \frac{z}{y^2}. \quad (4.24)$$

We then rewrite $\Phi(x_1, x_2, y)$ and $V_I(x_1, x_2, y)$ as deviations from the reference $AdS_5 \times S^5$ solution

$$\begin{aligned} \Phi &= \Phi^{(0)} + \Delta\Phi, & \Phi^{(0)} &\equiv \frac{z^{(0)}}{y^2}, \\ V &= V^{(0)} + \Delta V. \end{aligned} \quad (4.25)$$

It turns out that the most useful reference metric, around which the fluctuations are minimal, is the one centered around the center of mass defined by¹⁹

$$\vec{x}^{(0)} = \sum_{l=1}^M y_l^2 \vec{x}_l, \quad \sum_{l=1}^M y_l^2 = 1. \quad (4.26)$$

Furthermore, to simplify the calculations, henceforth we take that $\vec{x}^{(0)} = 0$.

Now, since the function $\Delta\Phi$ solves the six dimensional Laplace equation with $SO(4)$ symmetry, all the fluctuations can be expanded in a basis of $SO(4)$ invariant harmonics $Y^{\Delta,k}$ of $SO(6)$ (see appendix A for a summary of their properties). Therefore

$$\Delta\Phi = \sum_{\Delta \geq 2} \sum_{k=-\Delta, -\Delta+2, \dots}^{\Delta} \Delta\Phi_{\Delta,k} \frac{Y^{\Delta,-k}(\theta, \phi)}{R^{\Delta+4}}, \quad (4.27)$$

where θ, ϕ are coordinates on the S^5 (4.22). The $\Delta = 1$ term in (4.27) vanishes in the center of mass frame which we have adopted.

By doing the calculation in the center of mass frame (4.26) with $\vec{x}^{(0)} = 0$, we get from the M particle solution (4.5), (4.6) that²⁰

$$\begin{aligned} \Delta\Phi_{2,0} &= 4\sqrt{3} \sum_{l=1}^M \left(\vec{x}_l^2 - \frac{y_l^2 - 1}{2} \right) y_l^2 \\ \Delta\Phi_{2,\pm 2} &= 6e^{\mp 2i\psi} \sum_{l=1}^M (x_{l,1} \pm ix_{l,2})^2 y_l^2 \\ \Delta\Phi_{3,\pm 1} &= 8\sqrt{3}e^{\mp i\psi} \sum_{l=1}^M (x_{l,1} \pm ix_{l,2}) \left(\vec{x}_l^2 - y_l^2 + \frac{1}{2} \right) y_l^2 \\ \Delta\Phi_{3,\pm 3} &= 8\sqrt{2}e^{\mp i3\psi} \sum_{l=1}^M (x_{l,1} \pm ix_{l,2})^3 y_l^2. \end{aligned} \quad (4.28)$$

¹⁹Recall (4.9) that $y_l^2 = N_l/N$, and is therefore a natural weight for the point \vec{x}_l .

²⁰These expression will also be valid away from the center of mass frame, with a subtlety appearing only in $\Delta\Phi_{3,\pm 1}$, where the factor of $1/2$ needed to be determined by a separate calculation.

Now to calculate the desired expectation values of $\mathcal{O}_{\Delta,k}$, J_m^a and T_{mn} we use the relation (4.17), (4.18) between the expectation values and the fluctuations $[\hat{s}^{2I}]_2$, $[\hat{s}^{3I}]_3$, $[A_\mu^a]_2$ and $g_{(\cdot)mn}$. These in turn are related to the moments of $\Delta\Phi$ which we have just calculated by following the procedure in section 4.2 as we outline now.

We need to find the asymptotic form of the metric and five-form and then expand it in spherical harmonics as in (4.10), (4.11). This is done by plugging the coordinate transformation (4.20) into the M particle solution (4.5), (4.6), (4.7) and expanding for large R . In the multi-particle case this will not yield a manifestly asymptotically $AdS_5 \times S^5$ metric. This can be fixed by modifying the one-form V (4.7). Recall that V is determined from z by solving the equation $y dV = *_X dz$ and hence is only determined up to an exact form. In order to make manifest the $AdS_5 \times S^5$ asymptotics we add to V in (4.7) an exact one form

$$V \rightarrow V + d\omega. \quad (4.29)$$

The explicit expression for ω that yields the desired asymptotics is given in appendix D.

Now we can compute the various fluctuation modes. The warp factors in the metric (4.1) can be parametrized as follows

$$\begin{aligned} y\sqrt{\frac{2z-1}{2z+1}} &\equiv \cos^2\theta(1+A), & y\sqrt{\frac{2z+1}{2z-1}} &\equiv (R^2+1)(1+B), \\ \frac{2y}{\sqrt{4z^2-1}} &\equiv (R^2+\sin^2\theta)(1+C), & \frac{\sqrt{4z^2-1}}{2y} &\equiv \frac{(1+D)}{(R^2+\sin^2\theta)}. \end{aligned} \quad (4.30)$$

The functions A, B, C, D encode the deviations of the metric away from $AdS_5 \times S^5$. We express them in terms of $\Delta\Phi$, which scales (in the center of mass frame) like $1/R^6$ for large R (4.27). Up to $\mathcal{O}(1/R^4)$ they are given by

$$\begin{aligned} A &= \frac{1}{2}R^4\Delta\Phi + R^2\sin^2\theta\Delta\Phi - \frac{1}{8}R^8(\Delta\Phi)^2 \\ B &= -\frac{1}{2}R^4\Delta\Phi - R^2\sin^2\theta\Delta\Phi + \frac{3}{8}R^8(\Delta\Phi)^2 \\ C &= -\frac{1}{2}R^4\Delta\Phi - R^2\Delta\Phi + \frac{3}{8}R^8(\Delta\Phi)^2 \\ D &= \frac{1}{2}R^4\Delta\Phi + R^2\Delta\Phi - \frac{1}{8}R^8(\Delta\Phi)^2. \end{aligned} \quad (4.31)$$

We also expand the five-form flux (4.2) in a large R expansion. The expression for $\Delta\Phi$ is obtained from the explicit expression for the M particle solution (4.5), (4.6).

We now need to use (4.10), (4.11) to calculate the fluctuation modes that appear in (4.17), (4.18) which are needed for the computation of our correlation functions. Recalling the expression for the gauge invariant fluctuations (4.13), we finally have that

$$\begin{aligned} [\hat{s}^{2,k}]_2 &= \frac{1}{8}(\Delta\Phi)_{2,k}, & k &= 2, 0, -2 \\ [\hat{s}^{3,k}]_3 &= \frac{1}{12}(\Delta\Phi)_{3,k}, & k &= 3, 1, -1, -3 \\ [\hat{A}_\mu^\phi]_1 &= -\frac{3}{\sqrt{6}}(\Delta\Phi)_{2,0}. \end{aligned} \quad (4.32)$$

Now we recall the expressions for the moments of $\Delta\Phi$, which we calculated in (4.28) and can immediately write down the supergravity answer for the correlator of a surface operator \mathcal{O}_Σ with the CPO's $\mathcal{O}_{\Delta,k}$ (2.29) with $\Delta = 2, 3$. Writing the answer in gauge theory variables using (4.9) we get that

$$\begin{aligned}\frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{2,0} \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{|z|^2} \frac{8\pi^2}{\sqrt{6}\lambda} \left(\sum_{l=1}^M N_l \left((\beta_l^2 + \gamma_l^2) + \frac{\lambda}{4\pi^2} \frac{N - N_l}{2N} \right) \right), \\ \frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{2,2} \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{z^2} \frac{4\pi^2}{\sqrt{2}\lambda} \sum_{l=1}^M N_l (\beta_l + i\gamma_l)^2, \\ \frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{3,1} \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{z|z|^2} \frac{8\pi^3}{\sqrt{2}\lambda^{3/2}} \left(\sum_{l=1}^M N_l \left((\beta_l^2 + \gamma_l^2) + \frac{\lambda}{4\pi^2} \frac{N - 2N_l}{2N} \right) (\beta_l + i\gamma_l) \right), \\ \frac{\langle \mathcal{O}_\Sigma \cdot \mathcal{O}_{3,3} \rangle}{\langle \mathcal{O}_\Sigma \rangle} &= \frac{1}{z^3} \frac{8\pi^3}{\sqrt{3}\lambda^{3/2}} \sum_{l=1}^M N_l (\beta_l + i\gamma_l)^3.\end{aligned}\tag{4.33}$$

The remaining correlators can be obtained by complex conjugating (4.33).

To compare with the probe brane calculation (3.43) in the previous section, we consider the result of the supergravity calculation (4.33) for $N_1 = 1$ and $N_2 = N - 1$. Then we impose the center of mass condition, which is valid for the probe calculation and finally take $N \rightarrow \infty$ limit. The results are in precise agreement.

We can compare with the gauge theory for arbitrary surface operator parameters (N_l, β_l, γ_l) . As in the case of the probe calculation (3.43), the supergravity computation matches exactly the gauge theory result (2.30) for operators with $\Delta = |k|$ and the semiclassical gauge theory result in (2.30) exactly matches the leading term in the supergravity computation (4.33) for the rest of the chiral primary operators. The supergravity computation suggest that the loop corrections to the gauge theory result for the correlator between \mathcal{O}_Σ and $\mathcal{O}_{\Delta,k}$ truncate at order $\lambda^{(\Delta-|k|)/2}$.²¹

The vacuum expectation value of the U(1) current J_m^ψ is given by²²

$$\langle J_\psi^\psi \rangle_{\mathcal{O}_\Sigma} = \frac{2N}{\lambda} \sum_{l=1}^M \left(N_l (\beta_l^2 + \gamma_l^2) + \frac{\lambda}{4\pi^2} \frac{N - N_l}{2N} \right).\tag{4.34}$$

The first term agrees precisely with the gauge theory result (2.36) and the second captures the one loop correction.

We now proceed to calculate the correlator of \mathcal{O}_Σ with the stress-energy tensor T_{mn} . We first calculate the near boundary expansion of the metric along the AdS_5 directions. It

²¹This conjecture further requires that there are no $1/\sqrt{\lambda}$ correction for these observables in supergravity.

²²In computing the correlator with J_m^ψ we have used that the supergravity solution is U(1) invariant and that this U(1) is the diagonal sum of the $U(1) \subset SO(1,5)$ generated by J_m^ψ which rotates the space transverse to Σ and the $U(1) \subset SO(6)$ subgroup of the R -symmetry generated by J_m^ψ . The U(1) symmetry implies that $\langle J_m^\psi \rangle_{\mathcal{O}_\Sigma} = \langle J_m^\phi \rangle_{\mathcal{O}_\Sigma}$.

is given by

$$\begin{aligned}
 ds^2 = d\psi^2 & \left(R^2 + \frac{\Delta\Phi_{2,0}}{4\sqrt{3}R^2} + \frac{(\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{6 \times 24R^2} \right) + \frac{dR^2}{R^2 + 1} \left(1 - \frac{(\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{3 \times 24R^4} \right) \\
 & + ds_{AdS_3}^2 \left(R^2 + 1 - \frac{\Delta\Phi_{2,0}}{12\sqrt{3}R^2} + \frac{(\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{6 \times 24R^2} \right), \quad (4.35)
 \end{aligned}$$

where we sum over the $U(1)$ charge $k = 2, 0, -2$.

The correlator of \mathcal{O}_Σ with T_{mn} can be obtained from formula (4.18). This requires bringing the ten dimensional metric into the Fefferman-Graham form (4.15). In order to bring the metric (4.35) into this form we must perform the following change of coordinates

$$\begin{aligned}
 U &= \frac{1}{R} \left(1 - \frac{1}{4R^2} + \frac{1}{R^4} \left(\frac{1}{8} - \frac{(\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{24 \times 24} \right) \right) \\
 R &= \frac{1}{U} \left(1 - \frac{1}{4}U^2 - \frac{(\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{24 \times 24}U^4 \right). \quad (4.36)
 \end{aligned}$$

In the Fefferman-Graham gauge the metric (4.35) is given by

$$\begin{aligned}
 ds^2 = d\psi^2 & \left(\frac{1}{U^2} - \frac{1}{2} + \frac{18 + 24\sqrt{3}\Delta\Phi_{2,0} + (\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{288}U^2 \right) + \frac{dU^2}{U^2} \\
 & + ds_{AdS_3}^2 \left(\frac{1}{U^2} + \frac{1}{2} + \frac{18 - 8\sqrt{3}\Delta\Phi_{2,0} + (\Delta\Phi_{2,k}\Delta\Phi_{2,-k})}{288}U^2 \right). \quad (4.37)
 \end{aligned}$$

Now we can read off from (4.37) the various terms in the metric ($g_{(4)}$, $g_{(2)}$, $g_{(0)}$) that enter in the formula of the correlator of \mathcal{O}_Σ with the stress-energy tensor T_{mn} (4.18). The correlator is given by

$$\begin{aligned}
 \langle T_{\psi\psi} \rangle_{\mathcal{O}_\Sigma} &= \frac{N^2}{2\pi^2} \left(-\frac{3}{16} + \frac{1}{4\sqrt{3}}\Delta\Phi_{2,0} \right), \\
 \langle T_{ab} \rangle_{\mathcal{O}_\Sigma} &= \frac{N^2}{2\pi^2} \left(\frac{1}{16} - \frac{1}{12\sqrt{3}}\Delta\Phi_{2,0} \right) g_{ab}, \quad (4.38)
 \end{aligned}$$

where g_{ab} is an AdS_3 metric and the explicit expression for $\Delta\Phi_{2,0}$ is given in (4.28). More explicitly

$$\begin{aligned}
 \langle T_{\psi\psi} \rangle_{\mathcal{O}_\Sigma} &= -\frac{3N^2}{32\pi^2} + \frac{2}{g_{YM}^2} \sum_{l=1}^M N_l \left((\beta_l^2 + \gamma_l^2) + \frac{\lambda}{4\pi^2} \frac{N - N_l}{2N} \right) \\
 \langle T_{ab} \rangle_{\mathcal{O}_\Sigma} &= \left(\frac{N^2}{32\pi^2} - \frac{2}{3g_{YM}^2} \sum_{l=1}^M N_l \left((\beta_l^2 + \gamma_l^2) + \frac{\lambda}{4\pi^2} \frac{N - N_l}{2N} \right) \right) g_{ab}. \quad (4.39)
 \end{aligned}$$

The λ independent term encodes the Casimir energy in the gauge theory. The leading term in λ precisely agrees with the gauge theory calculation in the semiclassical regime (2.35). The last term can be interpreted as the one loop correction to the gauge theory result (2.35), and suggests that the correlator of \mathcal{O}_Σ with the stress-energy tensor T_{mn} receives only a one loop correction.

We note that the trace is $\langle T_m^m \rangle_{\mathcal{O}_\Sigma} = 0$, which reproduces the fact that the conformal anomaly of $\mathcal{N} = 4$ SYM on $AdS_3 \times S^1$ vanishes.

4.4 Correlator with Wilson and 't Hooft loop

We turn now to calculating the Wilson and 't Hooft loop expectation value in the supergravity background describing a surface operator \mathcal{O}_Σ . As usual in the holographic dual, the Wilson loop is described by a classical string [37, 38]. As in the calculation in the probe approximation in section 3.4, we have to identify the boundary conditions on the string stemming from the parameters of the Wilson loop W_{θ_0, ψ_0} (2.38) and 't Hooft loop T_{θ_0, ψ_0} . Unlike the probe approximation, here the string has no D3-branes to end on, rather it will wrap a non-trivial cycle in spacetime and close smoothly on itself, as we explain shortly.

To understand the boundary conditions we look at the asymptotic form of the metric (4.20), (4.22) and impose the same conditions as in the probe calculation in section 3.4. At the boundary, where $R \rightarrow \infty$, the string should wrap a circle of S^5 at $\theta = \theta_0$, so the coordinates r and y in (4.20) should diverge with a fixed ratio $r/y \rightarrow \tan \theta_0$. The string wraps circles both in AdS_5 and on S^5 , parametrized by ψ and ϕ with a relative phase $\alpha = \psi + \phi = \psi_0$, corresponding to that of the Wilson loop W_{θ_0, ψ_0} (2.38).

To get a smooth string solution with these boundary conditions, it has to have the topology of the disc and close smoothly on itself. Recall that the χ circle shrinks to zero size at each of the sources (\vec{x}_l, y_l) , so the string should extend from the boundary to one of the sources.

We take the ansatz where only χ depends on τ

$$x_i = x_i(\sigma), \quad y = y(\sigma), \quad \chi = \tau + \eta(\sigma). \quad (4.40)$$

Using the metric (4.1), the string action in the conformal gauge is

$$\mathcal{S} = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left(\frac{2y}{\sqrt{4z^2 - 1}} ((\eta' + V_\sigma)^2 + 1) + \frac{\sqrt{4z^2 - 1}}{2y} (y'^2 + x'_i x'_i) \right), \quad (4.41)$$

where V_σ is the pullback of the one-form V in the σ direction.

There are two Virasoro constraints for the diagonal and off-diagonal pieces in the stress-energy tensor

$$\begin{aligned} \eta' + V_\sigma &= 0 \\ y'^2 + x'_i x'_i &= \frac{4y^2}{4z^2 - 1}. \end{aligned} \quad (4.42)$$

The first equation will give η , once we solve for V (which depends only on (\vec{x}, y)). Then plugging the second Virasoro constraint into the equations of motion for the x_i and y one finds

$$\left(\frac{\sqrt{4z^2 - 1}}{2y} x'_1 \right)' = \left(\frac{\sqrt{4z^2 - 1}}{2y} x'_2 \right)' = \left(\frac{\sqrt{4z^2 - 1}}{2y} y' \right)' = 0. \quad (4.43)$$

These equations mean that the ratios of y' , x'_1 and x'_2 are constant along the worldsheet. Therefore the string is given by a straight line in this space!

These equations are actually unnecessarily complicated, since they are in the conformal gauge. We can switch to the Nambu-Goto formulation, the string action is very simple

$$\mathcal{S} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \sqrt{y'^2 + x'_i x'_i}, \quad (4.44)$$

In this formulation η does not appear and can be set to zero. The action is clearly just the length functional in (\vec{x}, y) space, which is solved by straight lines. These worldsheets correspond to the disks spanned by the straight lines in figure 2.

As mentioned above, the solution will have to extend from the boundary of spacetime to one of the sources, where the χ circle shrinks away. The length of the ray emanating from the source (\vec{x}_l, y_l) in the direction of the boundary point

$$(R \sin \theta_0 \cos \psi_0, R \sin \theta_0 \sin \psi_0, R \cos \theta_0), \quad (4.45)$$

with $R \rightarrow \infty$ is given by

$$R - y_l \cos \theta_0 - \sin \theta_0 (x_{l,1} \cos \psi_0 + x_{l,2} \sin \psi_0). \quad (4.46)$$

Recall that the “bubbling” supergravity background also has an NS-NS two-form gauge field so in addition to the Nambu-Goto term, the action includes the flux through the worldsheet, which according to (4.9), for a string ending on the source l is $-2\pi\alpha_l$.

After integrating over τ and removing the usual infrared divergence, which here is proportional to the cutoff R , the total action is

$$\mathcal{S} = - (y_l \cos \theta_0 + \sin \theta_0 (x_{l,1} \cos \psi_0 + x_{l,2} \sin \psi_0) + 2\pi i \alpha_l). \quad (4.47)$$

Finally we use (4.9) to rewrite the result in terms of the parameters from the gauge theory description, and sum over the different classical solutions, one for each source to get

$$\frac{\langle \mathcal{O}_\Sigma \cdot W_{\theta_0, \psi_0} \rangle}{\langle \mathcal{O}_\Sigma \rangle} \simeq \sum_{l=1}^M \exp \left[\sqrt{\frac{\lambda N_l}{N}} \cos \theta_0 + 2\pi \sin \theta_0 (\beta_l \cos \psi_0 + \gamma \sin \psi_0) + 2\pi i \alpha_l \right]. \quad (4.48)$$

As was mentioned in section 2.4, when expanding the exact result of the matrix model that captures the supersymmetric Wilson loops one finds two saddle points, one with a positive exponent of $\sqrt{\lambda}$ and one with a negative one. In [23] this was reproduced from a second string solution, an unstable one. Such solutions exist also in our case. The reason is that the (\vec{x}, y) space is restricted to $y \geq 0$, but the plane at $y = 0$ is not a boundary. In fact, if we consider a string with the ansatz (4.40), it would simply be reflected from the $y = 0$ plane.

We therefore have to amend (4.46). There isn't a unique geodesic from the boundary point (4.45) to each of the sources (\vec{x}_l, y_l) , but two. The second one is reflected through the $y = 0$ plane and its divergent length is²³

$$R + y_l \cos \theta_0 - \sin \theta_0 (x_{l,1} \cos \psi_0 + x_{l,2} \sin \psi_0). \quad (4.49)$$

²³This is immediate to derive by using the method of images with a source at the point $(\vec{x}_l, -y_l)$.

In addition to (4.48) we therefore find another sum coming from the other saddle points, which should all be unstable

$$\frac{\langle \mathcal{O}_\Sigma \cdot W_{\theta_0, \psi_0} \rangle}{\langle \mathcal{O}_\Sigma \rangle} \simeq \sum_{l=1}^M \exp \left[-\sqrt{\frac{\lambda N_l}{N}} \cos \theta_0 + 2\pi \sin \theta_0 (\beta_l \cos \psi_0 + \gamma \sin \psi_0) + 2\pi i \alpha_l \right]. \quad (4.50)$$

Note that the sign change is only on the first term in the exponent.

The sum of (4.48) and (4.50) is in precise agreement with the result of the gauge theory calculation (2.41) including our expectation for the term coming from self-contractions of the extra scalar ϕ^1 . We could not reproduce the prefactors which would require a string one-loop calculation to determine.

For an 't Hooft loop T_{θ_0, ψ_0} we get the same action, except that the string tension is multiplied by a factor of $1/g_s = 4\pi/g_{\text{YM}}^2$ and instead of coupling to the NS-NS field it gets a term from the RR field. The correlator is

$$\frac{\langle \mathcal{O}_\Sigma \cdot T_{\theta_0, \psi_0} \rangle}{\langle \mathcal{O}_\Sigma \rangle} \simeq \sum_{l=1}^M \exp \left[4\pi \sqrt{\frac{N N_l}{\lambda}} \cos \theta_0 + \frac{8\pi^2}{g_{\text{YM}}^2} (\beta_l \cos \psi_0 + \gamma \sin \psi_0) \sin \theta_0 + 2\pi i \eta_l \right], \quad (4.51)$$

to which again we should add the contributions of the unstable saddle points. This result is S -dual to (4.48), accounting for $\alpha_l \rightarrow \eta_l$ and also matches the general expectations for the behavior of the 't Hooft loop as discussed in section 2.

5. Discussion

In this paper we have began to study the properties of surface operators \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM performing various explicit calculations with them. We have studied the properties of surface operators \mathcal{O}_Σ using three different realizations, consisting of the gauge theory description, the probe D3-brane description and the “bubbling” supergravity description. We have found that a rather nice story emerges between these different realizations, allowing for precise matching between quantities computed using different descriptions.

For the expectation value of the surface operator with $\Sigma = R^2$ we found from the gauge theory and from the probe approximation that the expectation value is unity. It would be interesting to learn how to perform this calculation in the framework of the “bubbling” solutions, but it would probably be more advantageous to do it in the case of the surface operators in the six dimensional theory dual to M-theory on $AdS_7 \times S^4$ [47], since unlike our case, those operators do suffer from conformal anomalies.

For the correlator of a surface operator \mathcal{O}_Σ with a chiral primary operator $\mathcal{O}_{\Delta, k}$ and with a Wilson loop W_{θ_0, ψ_0} we did the calculation in the three different realizations of the surface operator \mathcal{O}_Σ and found remarkable agreement. The leading behavior agreed between all three regimes. Particularly exciting is the case of the Wilson loop, whose expectation value depends in general on the parameters of the surface operator $(N_l, \beta_l, \gamma_l, \alpha_l)$, those in the Wilson loop (θ_0, ψ_0) and the 't Hooft coupling λ .

In all these cases we restricted to a purely classical calculation on the gauge theory side, but we expect that those correlators will receive some quantum corrections. Remarkably,

the computations performed in the probe approximation (3.43) and using the “bubbling” supergravity description precisely reproduce the semiclassical gauge theory result (2.30). Moreover, we have found that the correlator of \mathcal{O}_Σ with some local operators have extra terms in (3.43), (4.33) which are of higher order in the ’t Hooft coupling compared to the semiclassical gauge theory result (2.30), and that nicely organize into a perturbative expansion in λ . Given these results, our expectation is that there are perturbative corrections to (2.30) which, nevertheless, terminate at finite order in the expansion in λ .²⁴ In particular, our results suggest that the correlator of \mathcal{O}_Σ with a chiral primary operator $\mathcal{O}_{\Delta,k}$ terminate at order²⁵ $\lambda^{(\Delta-|k|)/2}$, reproducing (3.43), (4.33). It would be interesting to reproduce this structure of the corrections directly from the path integral description of surface operators \mathcal{O}_Σ .

In the case of the Wilson loop W_{θ_0,ψ_0} we saw the appearance of terms of the form $\pm\sqrt{\lambda N_l/N}$ (4.48), which we attribute to the extra scalar ϕ^1 in (2.38). Again, we did not pursue the exact perturbative calculation of its effect, but we anticipate that it will give some modification of the matrix model of [24] and [25], perhaps along the lines of [26–28].

There exists a definition for the gauge theory in the presence of the surface operator \mathcal{O}_Σ also beyond the classical level. The path integral has to be done in the presence of the prescribed singularity. But concrete calculations have never been performed with that prescription (prior to this paper no calculations have been done even in the classical limit). Our results raise the prospects of being able to do such calculations in practice and in the supersymmetric cases considered in this paper possibly improving on our results and computing all quantum corrections. It would be interesting to use localization techniques to study whether the path integral of $\mathcal{N} = 4$ SYM in the presence of a surface operator \mathcal{O}_Σ becomes that of a reduced model, extending the recent result of Pestun [48], who has derived the matrix model in [24, 25] by analyzing the $\mathcal{N} = 4$ SYM path integral in the presence of a supersymmetric circular Wilson loop.

The fact that a supersymmetric surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM has multiple realizations where calculations can be done with precise agreement has an analog in the case of supersymmetric Wilson loops. These can be described in a variety of ways, perturbatively in $\mathcal{N} = 4$ SYM [24, 25], as strings in AdS_5 [37, 38, 49, 10], as a configuration of D3-branes [50–52] or as a configuration of D5-branes [53, 51], and finally as asymptotically $AdS_5 \times S^5$ “bubbling” supergravity backgrounds [54–56]. This paper aspires to be a first step in generalizing the impressive series of comparisons done for Wilson loops between gauge theory calculations and probe calculations. Much dynamical information can also be extracted from the “bubbling” geometries for Wilson loops [57, 58], extending the results obtained for Wilson loops

in the context of topological string theory [59] using “bubbling” Calabi-Yau’s [60, 61].

It would be interesting to perform similar computations to the ones in this paper for the surface operators of order type constructed in [2]. One can also consider maximally supersymmetric domain walls in $\mathcal{N} = 4$ SYM. For these the gauge theory realization is

²⁴This conjecture further requires that there are no $1/\sqrt{\lambda}$ correction for these observables in supergravity.

²⁵Note that $\Delta - |k|$ is always even, so we get a standard perturbation expansion in the ’t Hooft coupling λ .

given in terms of the low energy limit of the Hanany-Witten like setup involving D3, D5 and NS5 branes [62], the probe brane description is given by the branes in $AdS_5 \times S^5$ found by Karch and Randall [63], and the “bubbling” description is given by the smooth $OSP(4|4)$ invariant supergravity solutions found in [64, 55, 65]. The gauge theory description of supersymmetric domain walls in $\mathcal{N} = 4$ has been considered in [66–68], and a recent systematic study of supersymmetric boundary conditions has appeared in [69, 70].

The maximally supersymmetric operators in $\mathcal{N} = 4$ SYM are of great interest. Unlike the vacuum, there are many of them, with their classification still not complete. Yet, they possess a large degree of supersymmetry, sixteen supercharges, and therefore their dynamics are under good control, in some cases to all orders in perturbation theory.

The maximally supersymmetric local operators in $\mathcal{N} = 4$ SYM are captured by a reduction from the four dimensional gauge theory to a matrix quantum mechanics of a single Hermitean matrix [71], and are classified in terms of Young tableau. This matrix model can then be solved in terms of free fermions whose phase space has been identified by LLM in terms of boundary conditions for the “bubbling” supergravity description [8].

The maximally supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM are also of great interest and unlike the case of local supersymmetric operators, circular Wilson loops get contributions from all orders in perturbation theory. In fact the circular Wilson loop in $\mathcal{N} = 4$ SYM is captured by a zero-dimensional matrix model [24, 25], a result that has been derived recently using localization [48]. This matrix model reproduces correctly the expectation value of the loop as calculated by strings, D-branes [50, 53, 72] and also reproduces

properties [73] of the “bubbling” supergravity description [57, 58]. Modifications of the matrix model also encode the interaction of Wilson loops with local operators [28].

It is tempting therefore to try to see if there exists a matrix model that captures the essential properties of the maximally supersymmetric surface operators. We speculate in the following paragraphs over a possible candidate. It would be interesting to use localization to derive the reduced model capturing the properties of maximally supersymmetric surface operators in $\mathcal{N} = 4$ SYM.

Let us stress from the onset that it is not completely clear what properties this matrix model should capture. In the case of the supersymmetric circular Wilson loop, it calculates its expectation value, which may translate to some conformal anomaly in our case. Alas we saw that the maximally supersymmetric surface operators \mathcal{O}_Σ do not have a conformal anomaly. But one property that we do expect it to capture is the classification of surface operators in terms of a Levi group $L = \sum_{l=1}^M U(N_l) \subset U(N)$ and the parameters $(\alpha_l, \beta_l, \gamma_l, \eta_l)$ with $l = 1, \dots, M$.

One can consider a zero-dimensional complex matrix model with only a commutator-squared action

$$Z = \int \mathcal{D}\Phi \exp \left(\frac{1}{g_{\text{YM}}^2} \text{Tr}[\Phi, \bar{\Phi}]^2 \right). \quad (5.1)$$

A vacuum of this matrix model is given by a matrix that commute with its complex conjugate. Such a matrix can then be diagonalized with eigenvalues $\beta_l + i\gamma_l$, and the remaining symmetry, which depends on the degeneracy of those eigenvalues, would be exactly such a Levi group $L = \sum_{l=1}^M U(N_l)$.

This matrix model has an interesting feature. Due to the lack of a quadratic term in the action, when the path integral (5.1) is written in terms of its eigenvalues including the proper gauge fixing, the usual Van der Monde determinant is exactly cancelled by a similar contribution from the Fadeev-Popov determinant [74, 75].

Normally the eigenvalues are repelled from each-other due to the Van der Monde determinant, and this repulsion can then be captured by assigning the eigenvalues Fermi statistics. For our matrix model, there is no such eigenvalue repulsion and the vacua may well consist of a large number of coincident eigenvalues.

It is therefore natural to assign to the eigenvalues of the matrix model (5.1) Bose statistics, which nicely captures the structure of boundary conditions of the “bubbling” solutions for surface operators \mathcal{O}_Σ . The structure of the “bubbling” supergravity solutions for the surface operators \mathcal{O}_Σ is similar to that of the maximally supersymmetric local operators in $\mathcal{N} = 4$, where there is also a 3-dimensional space X , with coordinates (x_1, x_2, y) . The main difference between the “bubbling” solutions for local operators and surface operators is, however, in the boundary conditions for the Laplace equation (4.3), which determines the metric and five-form.

In the case of local operators the supergravity data which determines the solution is captured by binary information on the $y = 0$ plane, which is interpreted [8] as the occupation number of the free fermions. In our case, corresponding to maximally supersymmetric surface operators \mathcal{O}_Σ , the data which determines the solution is given by the position of point-like sources away from $y = 0$ (see figure 1). Projecting this picture on the $y = 0$ plane, it can be interpreted as a distribution of *bosons*. The distance in the y direction which represents the number of coincident D3-branes (or eigenvalues) through $y_l^2 = N_l/N$, is interpreted as their occupation number and is indeed quantized in terms of positive integers. The “bubbling” solution for local operators is encoded in terms of fermionic eigenvalues, while the “bubbling” solution for surface operators is encoded in terms of bosonic eigenvalues, a property captured by the matrix model (5.1).

One can enrich the matrix model (5.1) by inserting other observables in the integrand made out of the fields Φ and $\bar{\Phi}$, such as chiral primary operators. At least at the semi-classical level this matrix model captures the correlators of the surface operator \mathcal{O}_Σ with chiral primary operators.

While this picture is satisfyingly consistent, the matrix model (5.1) has some shortcomings in describing the surface operators \mathcal{O}_Σ . For example, it does not capture all the details of the vacuum structure. In particular, it does not capture the parameters α_l and η_l needed to fully characterize a surface operator \mathcal{O}_Σ , and that enter in the correlators of \mathcal{O}_Σ with the Wilson and 't Hooft operators.

One may be even more ambitious and hope to find a reduced model that captures the interactions between the different objects considered in this paper, including Wilson and 't Hooft loops. An example to keep in mind is the normal matrix model, which in [28] was shown to reproduce the interaction of maximally supersymmetric local operators and maximally supersymmetric Wilson loops.

A possible avenue is to replace the 0-dimensional matrix model (5.1) with matrix quantum mechanics, as is the case for the local operators. Given that instead of the theory

on $S^3 \times R$ we are dealing here with $\mathcal{N} = 4$ SYM on $AdS_3 \times S^1$, we propose to Wick-rotate and compactify the time direction. We propose to consider matrix quantum mechanics of a single complex matrix at finite temperature

$$L = \text{Tr} \left[\frac{1}{2} |D_\psi \Phi|^2 - \frac{1}{2} |\Phi|^2 + \frac{1}{4} [\Phi, \bar{\Phi}]^2 \right]. \quad (5.2)$$

Here the ψ direction is compact and the covariant derivative is $D_\psi \Phi = \partial_\psi \Phi - i[A_\psi, \Phi]$. In a vacuum of this theory, one can choose a gauge where A_ψ is diagonal, and it would then be characterized by the parameters α_l in (2.1) that label a surface operator \mathcal{O}_Σ .

In this matrix model it is easy to define, in addition to the local operators, a Wilson loop operator wrapping the ψ direction, which would be similar to (2.38), at least for $\theta_0 = \pi/2$. More degrees of freedom would be required in order to capture the dynamics of the extra scalar field ϕ^1 . We leave these speculations for future investigation.

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A. Spherical harmonics

In this appendix we summarize some of the properties of $SO(6)$ spherical harmonics (see [39] for more details). The scalar and vector $SO(6)$ spherical harmonics satisfy the following equations,

$$\begin{aligned} \Delta Y^{I_1} &= \Lambda^{I_1} Y^{I_1}, & \Lambda^{I_1} &= -\Delta(\Delta + 4), & \Delta &= 0, 1, 2, \dots \\ \Delta Y_a^{I_5} &= \Lambda^{I_5} Y_a^{I_5}, & \Lambda^{I_5} &= -(\Delta^2 + 4\Delta - 2), & \Delta &= 1, 2, \dots, \end{aligned} \quad (A.1)$$

where Δ is Laplacian on S^5 . They are normalized as follows

$$\begin{aligned} \int Y^{I_1} Y^{J_1} &= \pi^3 z(\Delta) \delta^{I_1 J_1}, \\ \int Y_a^{I_5} Y^{J_5 a} &= \pi^3 z(\Delta) \delta^{I_5 J_5}, \end{aligned} \quad (A.2)$$

where

$$z(\Delta) = \frac{1}{2^{\Delta-1}(\Delta+1)(\Delta+2)}. \quad (\text{A.3})$$

We consider the following metric (3.1) on S^5

$$ds^2 = \cos^2 \theta d\Omega_3 + d\theta^2 + \sin^2 \theta d\phi^2, \quad (\text{A.4})$$

where $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$.

The $\text{SO}(4)$ invariant scalar spherical harmonics of $\text{SO}(6)$ are given by

$$Y^{\Delta,k}(\theta, \phi) = c_{\Delta,k} y^{\Delta,k}(\theta) e^{ik\phi}, \quad (\text{A.5})$$

where Δ is a non-negative integer, $k \in (-\Delta, -\Delta+2, \dots, \Delta-2, \Delta)$, $c_{\Delta,k}$ is a normalization constant and $y^{\Delta,k}$ is given by a hypergeometric function

$$y^{\Delta,k} = \sin^{|k|} \theta {}_2F_1 \left(-\frac{1}{2}(\Delta - |k|), 2 + \frac{1}{2}(\Delta + |k|), 1 + |k|; \sin^2 \theta \right). \quad (\text{A.6})$$

For the computations in this paper, we need the following scalar spherical harmonics

$$\begin{aligned} Y^{2,0} &= \frac{1}{2\sqrt{3}}(3\sin^2 \theta - 1), \\ Y^{\Delta,\pm\Delta} &= \frac{1}{2^{\Delta/2}} \sin^{\Delta} \theta e^{\pm i\Delta\phi}, \\ Y^{3,\pm 1} &= \frac{\sqrt{3}}{4} \sin \theta (2\sin^2 \theta - 1) e^{\pm i\phi} \end{aligned} \quad (\text{A.7})$$

and vector spherical harmonic

$$Y^1 = \frac{1}{\sqrt{2}} \sin^2 \theta d\phi. \quad (\text{A.8})$$

In order to write down the explicit form of the dimension $\Delta = 2, 3$ $\text{SO}(4)$ invariant chiral primary operators (2.29) we write the spherical harmonics in (A.8) using the following embedding coordinates

$$\begin{aligned} X^i &= \cos \theta \Theta_i & i &= 1, \dots, 4 \\ X^5 &= \sin \theta \cos \phi, \\ X^6 &= \sin \theta \sin \phi, \end{aligned} \quad (\text{A.9})$$

where $\sum_{i=1}^4 \Theta_i^2 = 1$.

Evaluating the harmonics at $\theta = \pi/2$ we get that $Y^{\Delta,k}(\theta = \pi/2, \phi) = C_{\Delta,k} e^{ik\phi}$, where

$$C_{2,0} = 1/\sqrt{3}, \quad C_{2,\pm 2} = 1/2, \quad C_{3,\pm 1} = \sqrt{3}/4, \quad C_{3,\pm 3} = 1/2\sqrt{2} \quad (\text{A.10})$$

are obtained from the normalization condition (A.2).

B. Supersymmetry of Wilson loops

We review here the supersymmetry calculation for the Wilson loop (2.38), as done in [23] and show that they are compatible with those preserved by the surface operator \mathcal{O}_Σ . We follow the notations in the appendix of [7], using 10-dimensional gamma matrices such that the variation of the two components of the gauge field gives terms with the gamma matrices γ^2 and γ^3 and of the two scalars in Φ and $\bar{\Phi}$ gives $\gamma^8 \pm i\gamma^9$. In addition the variation of the extra scalar ϕ^1 gives γ^4 .

The supersymmetry variation of the Wilson loop W_{θ_0, ψ_0} (2.38) gives an expression proportional to

$$\begin{aligned} & \left(\sin \psi (-i\gamma^2 + \sin \theta_0 \sin \psi_0 \gamma^8 - \sin \theta_0 \cos \psi_0 \gamma^9) \right. \\ & \left. + (\cos \psi (i\gamma^3 + \sin \theta_0 \cos \psi_0 \gamma^8 + \sin \theta_0 \sin \psi_0 \gamma^9) + \cos \theta_0 \gamma^4) (\epsilon_1 + \cos \psi \gamma^2 \epsilon_2 + \sin \psi \gamma^3 \epsilon_2) \right). \end{aligned} \quad (\text{B.1})$$

Here ϵ_1 are the parameters of the Poincaré supersymmetry transformations and ϵ_2 are the parameters of the conformal supersymmetry transformations. The supersymmetries preserved by the loop are those where the above expression (B.1) vanishes for all ψ . One can therefore expand (B.1) in Fourier modes and get different equations for each Fourier component: 1, $\sin \psi$, $\cos \psi$, $\sin 2\psi$ and $\cos 2\psi$.

It turns out that requiring all these projectors to annihilate the spinors is a consistent set of equations which have 8 independent solution (for $\theta_0 = 0$ there are 16). Therefore such a Wilson loop is quarter BPS.

These equations in general relate ϵ_1 and ϵ_2 , but we can separate them and find the following equation that is satisfied by both spinors

$$\sin \theta_0 [\sin \psi_0 (\gamma^{28} + \gamma^{39}) + \cos \psi_0 (\gamma^{38} - \gamma^{20})] \epsilon = 0. \quad (\text{B.2})$$

The supersymmetries unbroken by the surface operator in the plane transverse to the loop are the solutions to the equations [7] (and see also the next appendix)

$$\gamma_{2389} \epsilon = -\epsilon, \quad (\text{B.3})$$

with either ϵ_1 or ϵ_2 . It is easy to check that any solution to (B.3) automatically solves (B.2). Therefore the presence of the surface operator \mathcal{O}_Σ does not break any of the supersymmetries left unbroken by the Wilson loop (2.38), they share eight supercharges.

In the special case of the maximally supersymmetric loop, for $\theta_0 = 0$, the surface operator does break half of the supersymmetries of the Wilson loop, so the combined system is quarter supersymmetric, i.e. preserving eight supercharges, like the generic case.

C. Supersymmetry of probe D3-brane

In this appendix, we study the supersymmetries left unbroken by a probe D3-brane corresponding to a surface operator \mathcal{O}_Σ . To avoid the complications of Euclidean supersymmetry, we consider \mathcal{O}_Σ supported on $\Sigma = R^{1,1}$ in $R^{1,3}$. We take the metric on $AdS_5 \times S^5$ to be

$$ds^2 = \frac{d\omega^2}{\omega^2} + \omega^2 (-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + d\theta_0^2 + \sum_{j=1}^3 \prod_{i=0}^{j-1} \sin^2 \theta_i d\theta_j^2 + \prod_{i=0}^4 \sin^2 \theta_i d\phi^2. \quad (\text{C.1})$$

The Killing spinor of $AdS_5 \times S^5$ in this coordinate system is given by [76]

$$\epsilon = \left[-\omega^{-\frac{1}{2}} \gamma_4 h(\theta_i, \phi) + \omega^{\frac{1}{2}} h(\theta_i, \phi) x_m \gamma^m \right] \eta_2 + \omega^{\frac{1}{2}} h(\phi_i, \phi) \eta_1, \quad (C.2)$$

where γ_μ are the tangent space gamma matrices and

$$h(\phi_i, \phi) = e^{\frac{1}{2}\theta_0\gamma_{45}} e^{\frac{1}{2}\theta_1\gamma_{56}} e^{\frac{1}{2}\theta_2\gamma_{67}} e^{\frac{1}{2}\theta_3\gamma_{78}} e^{\frac{1}{2}\phi\gamma_{89}}. \quad (C.3)$$

η_1 and η_2 are constant ten dimensional complex spinors that have positive and negative ten dimensional chirality respectively,

$$\gamma_{11}\eta_1 = -\eta_1, \quad \gamma_{11}\eta_2 = \eta_2, \quad (C.4)$$

which also satisfy

$$\eta_1 = (1 - i\gamma_{0123})\epsilon_1, \quad \eta_2 = (1 + i\gamma_{0123})\epsilon_2, \quad (C.5)$$

where ϵ_1 and ϵ_2 are ten dimensional Majorana-Weyl fermions with positive and negative chirality. At the boundary, ϵ_1 is identified with a Poincaré supersymmetry parameter and ϵ_2 with a conformal supersymmetry parameter of the dual gauge theory.

The embedding of the supersymmetric probe D3 brane (3.8) is given in this coordinate system by

$$r = \frac{\sinh u_0}{\omega}, \quad \theta_i = \frac{\pi}{2}, \quad \psi + \phi = \phi_0, \quad (C.6)$$

where $x_2 + ix_3 = r e^{i\psi}$. In this metric and embedding, the induced metric on the D3-brane is $AdS_3 \times S^1$

$$ds^2 = \cosh^2 u_0 \left(\frac{d\omega^2}{\omega^2} + \frac{\omega^2}{\cosh^2 u_0} (-dx_0^2 + dx_1^2) + d\phi^2 \right). \quad (C.7)$$

The supersymmetries that are preserved by the D3-brane are those that satisfy

$$\Gamma_\kappa \epsilon|_{D3} = \epsilon|_{D3}, \quad (C.8)$$

where ϵ is the Killing spinor of the $AdS_5 \times S^5$ background given in (C.2), and Γ_κ is the κ -symmetry projection matrix

$$\Gamma_\kappa = -i\mathcal{L}_{\text{DBI}}^{-1} \partial_0 x^\mu \partial_1 x^\nu \partial_\omega x^\rho \partial_\phi x^\sigma \Gamma_{\mu\nu\rho\sigma}. \quad (C.9)$$

For our D3 embedding (C.6)

$$\begin{aligned} i\mathcal{L}_{\text{DBI}}\Gamma_\kappa = & \Gamma_{01\omega\phi} + \frac{\sinh^2 u_0}{\omega^3} \Gamma_{0123} - \frac{\sinh u_0}{\omega^2} \sin(\phi - \phi_0)(\omega\Gamma_{01\omega 2} - \Gamma_{013\phi}) \\ & - \frac{\sinh u_0}{\omega^2} \cos(\phi - \phi_0)(\omega\Gamma_{01\omega 3} + \Gamma_{012\phi}). \end{aligned} \quad (C.10)$$

The gamma matrices appearing in the last two equations are related to the tangent space γ matrices by

$$\Gamma_\mu = \omega \gamma_\mu, \quad \Gamma_\omega = \frac{1}{\omega} \gamma_4, \quad \Gamma_\phi = \gamma_9. \quad (C.11)$$

Using $\mathcal{L}_{\text{DBI}} = \omega \cosh^2 u_0$ we may write Γ_κ as

$$\Gamma_\kappa = -\frac{i}{\cosh^2 u_0} \left((\gamma_{0149} + \sinh^2 u_0 \gamma_{0123}) - \sinh u_0 \sin(\phi - \phi_0)(\gamma_{0142} - \gamma_{0139}) \right. \\ \left. - \sinh u_0 \cos(\phi - \phi_0)(\gamma_{0143} + \gamma_{0129}) \right). \quad (\text{C.12})$$

After some algebra we get from (C.8)

$$(1 + \gamma_{2389})\epsilon_1 = 0, \\ (1 + \gamma_{2389})\epsilon_2 = 0. \quad (\text{C.13})$$

The preserved supersymmetries (C.13) are the same as those preserved by a maximally supersymmetric surface operator \mathcal{O}_Σ in $\mathcal{N} = 4$ SYM (see for e.g. [7]).

D. Gauge transformation on V

As mentioned in the main text, the one-form (4.7) is determined up to an exact form by solving the equation $dV = \frac{1}{y} *_X dz$. In order to get a manifestly asymptotically $AdS_5 \times S^5$ geometry, we need to add to V (4.7) an exact one-form

$$V \rightarrow V + d\omega. \quad (\text{D.1})$$

The exact form is given up to $\mathcal{O}(1/R^4)$

$$\omega = -\frac{M-1}{2}\alpha + \sum_{l=1}^M \left(\frac{x_{l,2} \cos \phi - x_{l,1} \sin \phi}{2 \sin \theta R} \right) - \sum_{l=1}^M \left(\frac{(x_{l,1}^2 - x_{l,2}^2) \sin 2\alpha - 2x_{l,1}x_{l,2} \cos 2\alpha}{4 \sin^2 \theta R^2} \right) \\ - \sum_{l=1}^M \left(\frac{x_{l,1}^3 \sin 3\alpha + x_{l,2}^3 \cos 3\alpha - 3x_{l,1}^2 x_{l,2} \cos 3\alpha - 3x_{l,1} x_{l,2}^2 \sin 3\alpha}{6 \sin^3 \theta R^3} \right) + \mathcal{O}\left(\frac{1}{R^4}\right). \quad (\text{D.2})$$

where we have used the coordinates in (4.20). In computing the correlation functions of surface operators \mathcal{O}_Σ with local operators using the “bubbling” supergravity solution, we have used this expression for V .

References

- [1] S. Gukov and E. Witten, *Gauge theory, ramification and the geometric Langlands program*, [hep-th/0612073](#).
- [2] E.I. Buchbinder, J. Gomis and F. Passerini, *Holographic gauge theories in background fields and surface operators*, *JHEP* **12** (2007) 101 [[arXiv:0710.5170](#)].
- [3] J.A. Harvey and A.B. Royston, *Localized modes at a D-brane-O-plane intersection and heterotic Alice strings*, *JHEP* **04** (2008) 018 [[arXiv:0709.1482](#)].
- [4] E. Witten, *Gauge theory and wild ramification*, [arXiv:0710.0631](#).
- [5] S. Gukov and E. Witten, *Rigid surface operators*, [arXiv:0804.1561](#).

- [6] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and exceptional branes in F-theory — I*, [arXiv:0802.3391](#).
- [7] J. Gomis and S. Matsuura, *Bubbling surface operators and S-duality*, *JHEP* **06** (2007) 025 [[arXiv:0704.1657](#)].
- [8] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [[hep-th/0409174](#)].
- [9] H. Lin and J.M. Maldacena, *Fivebranes from gauge theory*, *Phys. Rev. D* **74** (2006) 084014 [[hep-th/0509235](#)].
- [10] D.E. Berenstein, R. Corrado, W. Fischler and J.M. Maldacena, *The operator product expansion for Wilson loops and surfaces in the large- N limit*, *Phys. Rev. D* **59** (1999) 105023 [[hep-th/9809188](#)].
- [11] C.R. Graham and E. Witten, *Conformal anomaly of submanifold observables in AdS/CFT correspondence*, *Nucl. Phys. B* **546** (1999) 52 [[hep-th/9901021](#)].
- [12] M. Henningson and K. Skenderis, *Weyl anomaly for Wilson surfaces*, *JHEP* **06** (1999) 012 [[hep-th/9905163](#)].
- [13] A. Gustavsson, *On the Weyl anomaly of Wilson surfaces*, *JHEP* **12** (2003) 059 [[hep-th/0310037](#)].
- [14] A. Gustavsson, *Conformal anomaly of Wilson surface observables: a field theoretical computation*, *JHEP* **07** (2004) 074 [[hep-th/0404150](#)].
- [15] N. Drukker and B. Fiol, *On the integrability of Wilson loops in $AdS_5 \times S^5$: some periodic ansatze*, *JHEP* **01** (2006) 056 [[hep-th/0506058](#)].
- [16] R.M. Rohm, *Some current problems in particle physics beyond the standard model*, Ph.D. thesis, UMI-85-18643.
- [17] J. Preskill and L.M. Krauss, *Local discrete symmetry and quantum mechanical hair*, *Nucl. Phys. B* **341** (1990) 50.
- [18] L. Bonora, P. Pasti and M. Bregola, *Weyl cocycles*, *Class. and Quant. Grav.* **3** (1986) 635.
- [19] A. Schwimmer and S. Theisen, *Entanglement entropy, trace anomalies and holography*, *Nucl. Phys. B* **801** (2008) 1 [[arXiv:0802.1017](#)].
- [20] S.S. Gubser and I.R. Klebanov, *Absorption by branes and Schwinger terms in the world volume theory*, *Phys. Lett. B* **413** (1997) 41 [[hep-th/9708005](#)].
- [21] A. Kapustin, *Wilson-’t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev. D* **74** (2006) 025005 [[hep-th/0501015](#)].
- [22] K. Zarembo, *Supersymmetric Wilson loops*, *Nucl. Phys. B* **643** (2002) 157 [[hep-th/0205160](#)].
- [23] N. Drukker, *1/4 BPS circular loops, unstable world-sheet instantons and the matrix model*, *JHEP* **09** (2006) 004 [[hep-th/0605151](#)].
- [24] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **582** (2000) 155 [[hep-th/0003055](#)].
- [25] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [[hep-th/0010274](#)].

- [26] G.W. Semenoff and D. Young, *Exact 1/4 BPS loop: chiral primary correlator*, *Phys. Lett. B* **643** (2006) 195 [[hep-th/0609158](#)].
- [27] G.W. Semenoff and K. Zarembo, *More exact predictions of SUSYM for string theory*, *Nucl. Phys. B* **616** (2001) 34 [[hep-th/0106015](#)].
- [28] K. Okuyama and G.W. Semenoff, *Wilson loops in $N = 4$ SYM and fermion droplets*, *JHEP* **06** (2006) 057 [[hep-th/0604209](#)].
- [29] N.R. Constable, J. Erdmenger, Z. Guralnik and I. Kirsch, *Intersecting D3-branes and holography*, *Phys. Rev. D* **68** (2003) 106007 [[hep-th/0211222](#)].
- [30] C. Fefferman and R. Graham, *Conformal invariants*, *Astérisque* hors série (1995) pg. 95.
- [31] S. Giombi, R. Ricci and D. Trancanelli, *Operator product expansion of higher rank Wilson loops from D-branes and matrix models*, *JHEP* **10** (2006) 045 [[hep-th/0608077](#)].
- [32] B. Chen, C.-Y. Liu and J.-B. Wu, *Operator product expansion of Wilson surfaces from M5-branes*, *JHEP* **01** (2008) 007 [[arXiv:0711.2194](#)].
- [33] R. Corrado, B. Florea and R. McNees, *Correlation functions of operators and Wilson surfaces in the $D = 6$, $(0, 2)$ theory in the large- N limit*, *Phys. Rev. D* **60** (1999) 085011 [[hep-th/9902153](#)].
- [34] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [35] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The mass spectrum of chiral $N = 2$ $D = 10$ supergravity on S^5* , *Phys. Rev. D* **32** (1985) 389.
- [36] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large- N* , *Adv. Theor. Math. Phys.* **2** (1998) 697 [[hep-th/9806074](#)].
- [37] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large- N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J. C* **22** (2001) 379 [[hep-th/9803001](#)].
- [38] J.M. Maldacena, *Wilson loops in large- N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)].
- [39] K. Skenderis and M. Taylor, *Anatomy of bubbling solutions*, *JHEP* **09** (2007) 019 [[arXiv:0706.0216](#)].
- [40] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [41] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)].
- [42] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [[hep-th/0002230](#)].
- [43] M. Bianchi, D.Z. Freedman and K. Skenderis, *How to go with an RG flow*, *JHEP* **08** (2001) 041 [[hep-th/0105276](#)].
- [44] M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, *Nucl. Phys. B* **631** (2002) 159 [[hep-th/0112119](#)].

- [45] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. and Quant. Grav.* **19** (2002) 5849 [[hep-th/0209067](#)].
- [46] K. Skenderis and M. Taylor, *Kaluza-Klein holography*, *JHEP* **05** (2006) 057 [[hep-th/0603016](#)].
- [47] O. Lunin, *1/2-BPS states in M-theory and defects in the dual CFTs*, *JHEP* **10** (2007) 014 [[arXiv:0704.3442](#)].
- [48] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, [arXiv:0712.2824](#).
- [49] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev. D* **60** (1999) 125006 [[hep-th/9904191](#)].
- [50] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [[hep-th/0501109](#)].
- [51] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074 [[hep-th/0604007](#)].
- [52] J. Gomis and F. Passerini, *Wilson loops as D3-branes*, *JHEP* **01** (2007) 097 [[hep-th/0612022](#)].
- [53] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **05** (2006) 037 [[hep-th/0603208](#)].
- [54] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, *Int. J. Mod. Phys. A* **22** (2007) 1353 [[hep-th/0601089](#)].
- [55] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026 [[hep-th/0604133](#)].
- [56] E. D'Hoker, J. Estes and M. Gutperle, *Gravity duals of half-BPS Wilson loops*, *JHEP* **06** (2007) 063 [[arXiv:0705.1004](#)].
- [57] T. Okuda and D. Trancanelli, *Spectral curves, emergent geometry, and bubbling solutions for Wilson loops*, *JHEP* **09** (2008) 050 [[arXiv:0806.4191](#)].
- [58] J. Gomis, S. Matsuura, T. Okuda and D. Trancanelli, *Wilson loop correlators at strong coupling: from matrices to bubbling geometries*, *JHEP* **08** (2008) 068 [[arXiv:0807.3330](#)].
- [59] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys. B* **577** (2000) 419 [[hep-th/9912123](#)].
- [60] J. Gomis and T. Okuda, *Wilson loops, geometric transitions and bubbling Calabi-Yau's*, *JHEP* **02** (2007) 083 [[hep-th/0612190](#)].
- [61] J. Gomis and T. Okuda, *D-branes as a bubbling Calabi-Yau*, *JHEP* **07** (2007) 005 [[arXiv:0704.3080](#)].
- [62] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three- dimensional gauge dynamics*, *Nucl. Phys. B* **492** (1997) 152 [[hep-th/9611230](#)].
- [63] A. Karch and L. Randall, *Locally localized gravity*, *JHEP* **05** (2001) 008 [[hep-th/0011156](#)].
- [64] J. Gomis and C. Romelsberger, *Bubbling defect CFT's*, *JHEP* **08** (2006) 050 [[hep-th/0604155](#)].
- [65] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS Type IIB interface solutions I: local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [[arXiv:0705.0022](#)].

- [66] O. DeWolfe, D.Z. Freedman and H. Ooguri, *Holography and defect conformal field theories*, *Phys. Rev. D* **66** (2002) 025009 [[hep-th/0111135](#)].
- [67] A.B. Clark, D.Z. Freedman, A. Karch and M. Schnabl, *The dual of Janus ($(<:) \leftrightarrow (:>)$) an interface CFT*, *Phys. Rev. D* **71** (2005) 066003 [[hep-th/0407073](#)].
- [68] C. Kim, E. Koh and K.-M. Lee, *Janus and multifaced supersymmetric theories*, *JHEP* **06** (2008) 040 [[arXiv:0802.2143](#)].
- [69] D. Gaiotto and E. Witten, *Supersymmetric boundary conditions in $N = 4$ super Yang-Mills theory*, [arXiv:0804.2902](#).
- [70] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings and the theta-angle in $N = 4$ super Yang-Mills theory*, [arXiv:0804.2907](#).
- [71] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [[hep-th/0403110](#)].
- [72] S.A. Hartnoll and S.P. Kumar, *Higher rank Wilson loops from a matrix model*, *JHEP* **08** (2006) 026 [[hep-th/0605027](#)].
- [73] T. Okuda, *A prediction for bubbling geometries*, *JHEP* **01** (2008) 003 [[arXiv:0708.3393](#)].
- [74] V.A. Kazakov, I.K. Kostov and N.A. Nekrasov, *D-particles, matrix integrals and KP hierarchy*, *Nucl. Phys. B* **557** (1999) 413 [[hep-th/9810035](#)].
- [75] J. Hoppe, V. Kazakov and I.K. Kostov, *Dimensionally reduced SYM_4 as solvable matrix quantum mechanics*, *Nucl. Phys. B* **571** (2000) 479 [[hep-th/9907058](#)].
- [76] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, *JHEP* **06** (2002) 025 [[hep-th/0204054](#)].